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Effective equations in quantum cosmology

A.O. Barvinsky [a], D.V. Nesterov [b]

[a] *Theory Department, Lebedev Physics Institute and Lebedev Research Center in Physics, Leninsky Prospect 53, Moscow 117924, Russia*

[b] *Physics Department, Moscow State University, Moscow 117234, Russia*

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Abstract

We develop a general framework for effective equations of expectation values in quantum cosmology and pose for them the quantum Cauchy problem with no-boundary and tunneling wavefunctions. Cosmological configuration space is decomposed into two sectors that give qualitatively different contributions to the radiation currents in effective equations. The field-theoretical sector of inhomogeneous modes is treated by the method of Euclidean effective action, while the quantum mechanical sector of the spatially homogeneous inflaton is handled by the technique of manifest quantum reduction to gauge invariant cosmological perturbations. We apply this framework in the model with a big negative non-minimal coupling, which incorporates a recently proposed low energy (GUT scale) mechanism of the quantum origin of the inflationary Universe and study the effects of the quantum inflaton mode. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

This paper is a sequel to the previous work [1] on quantum dynamics of the early Universe starting from initial conditions inspired by quantum cosmology. These initial conditions, encoded in the no-boundary [2–4] and tunneling [5] quantum states, can be a source of the inflationary scenario at the low (typically GUT) energy scale [6,7], compatible with the observational status of inflation theory. In [6,7] the initial conditions were found as a sharp peak in the probability distribution of the quantum scalar field whose expectation value simulates the effective Hubble constant and drives inflation. The parameters of this peak — its location and quantum width — are suppressed relative to the Planckian values by a small factor coinciding with the magnitude of the CMBR anisotropy, $\Delta T/T \sim 10^{-5}$, according to normalization on COBE [8,9]. In [10,11] these results were extended to the open model based on the Hawking–Turok instanton [12].

E-mail address: barvin@td.lpi.ru (A.O. Barvinsky).

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Although, a priori these features make this model attractive, serious objections usually arise regarding the present day status of initial conditions in cosmology. Their issue in inflation theory is a subject of strong debate in current literature [13] and numerous meetings on structure formation of the Universe. The possibility of quantum cosmological imprint on the observational data was called in question from the viewpoint of the self-reproducing inflation scenario [14,15], eternal inflation [13] and anthropic principle [16]. These objections rely on special conditions guaranteeing self-reproduction and eternal inflation. However, in the model of [1,6,7] these conditions are not satisfied at the probability peak of the distribution function. The energy scale at this peak turns out to be far below the threshold beyond which eternal inflation begins (see Section 2 below). Thus this model represents a sound example of the low-energy phenomenon of the quantum birth of the Universe deserving further studies.

The dynamical consequences of the inflaton probability peak were first studied at the level of classical equations [6,7]. Then they were reexamined at the level of effective equations for quantum expectation values of fields [1]. Note that in cosmology it is often considered that the description in terms of the mean field and the set of higher-order correlation functions is not exhaustive. In contrast with conventional quantum field theory, interpretation of quantum cosmology is marred by the conceptual issues of the physical observer being the part of a quantum system. In particular, it is often assumed that the observational process leads to the reduction of the cosmological state to those values of fields which are far from their quantum expectations. This complicated and, in our opinion, not yet completely understood phenomenon is usually modelled in terms of stochastic random process serving, in particular, as a basis for eternal inflation and self-reproduction. Thus, eternal inflation starting from members of quantum ensemble remote from the probability peak can start dominating in the full stochastic ensemble which replaces the quantum one. This phenomenon goes beyond the scope of this paper, because we will be interested in the effective equations of motion for a particular, most probable, member of this ensemble. This can be justified by the fact that for a very sharp probability peak (located far away from the self-reproduction domain) stochasticity is unlikely to contribute essentially to effective equations in question. [1]

The effective equations analyses of [1] was not complete — their quantum contribution included all modes *except* the main spatially homogeneous mode related to quantum fluctuations of the inflaton field. This is the main spatially global degree of freedom

[1] Note that effective equations is a well defined concept when they are supplied with initial conditions given by expectation values in the initial quantum state. These equations can be derived from the demand that their solution is an evolving expectation value of the quantum Heisenberg operator (see Section 3 below). From this it unambiguously follows that for small quantum effects effective equations coincide with classical ones up to loop corrections. Similarly, the history evolving from the outskirts of the initial quantum distribution demands a precise definition. A priori ascribing to it classical equations of motion might lead to inconsistencies. Indeed, it is natural to assume that this history is again the time evolving expectation value, but now — with respect to the new quantum state which arose as a result of the wave function reduction to the domain distant from the probability maximum. Its equation of motion will be semiclassical only when this new state is semiclassical itself. As we see, this in turn implies strong assumptions about the nature of wave function collapse underlying the stochastic description and eternal inflation.

in cosmology. It is responsible for the quantum fluctuations of the homogeneous (minisuperspace) background on top of which all the other inhomogeneous modes dynamically evolve. As was understood in [1], the contribution of this mode requires the calculational technique different from the technique for inhomogeneous ones. The latter is strongly facilitated by the method of *Euclidean effective action* which is based on specific properties of their quantum state on the quasi-de Sitter background — Euclidean de Sitter invariant vacuum [17]. On the contrary, the quantum state of the inflaton is irrelevant to this vacuum and its contribution cannot be obtained by the analytic continuation from the Euclidean effective action. Rather, this state is determined by the peak of the inflaton probability distribution and can be approximated by the gaussian packet. Direct quantum averaging with respect to this packet is required for obtaining the contribution of the inflaton mode to effective equations. This step, however, should be preceded by several other calculational steps involving the Hamiltonian reduction to the physical sector, setting the quantum Cauchy problem in the minisuperspace sector of the system, etc. Thus, the goal of the present paper is to pose the quantum Cauchy problem in cosmology for the no-boundary and tunneling quantum states, derive the one-loop contribution of the homogeneous mode to effective equations and analyze its influence on the inflationary dynamics.

One of particular questions, to be clarified in this work, is to what an extent this contribution can change the conclusions of our previous papers [1,7]. In the spatially closed model with the no-boundary and tunneling cosmological states these conclusions were pretty stringent. The no-boundary quantum state can give rise only to the eternally long inflation, while the finite inflation stage with the conventional exit to the matter-dominated Universe exists only for the tunneling state. [2] These conclusions give an undoubtful preference to the tunneling state from the viewpoint of the observational cosmology, but, at the fundamental level, the tunneling state has an intrinsic problem if one goes beyond the tree-level approximation. [3] So, there is a hope that the inclusion of quantum fluctuations in the minisuperspace sector can render the no-boundary state viable from the viewpoint of the inflationary cosmology. Unfortunately, as we shall see, this hope does not realize — the effect of this mode turns out to be too small. This result, however, is model dependent, and the application of the general framework of this paper might lead to other conclusions in alternative models with well-defined quantum states of the inflaton.

It should be emphasized, that despite intensive studies on the inflationary dynamics, quantum nature of the homogeneous inflaton mode has not yet been completely under-

[2] In open cosmology based on the Hawking–Turok instanton [12] the initial conditions for finite inflation stage can be realized for the no-boundary state [10,11], but the models of [11,12] with all their merits and disadvantages go beyond the scope of this paper.

[3] Point is that the classical Euclidean action enters the exponential of the tunneling wavefunction with the "wrong" sign — opposite to that of the loop corrections [18]. This mismatch invalidates the conventional renormalization procedure of absorbing the ultraviolet divergences into the redefinition of classical coupling constants [19]. This inconsistency does not break the results of [1,6,7] (heavily relying on loop effects) for accidental reasons — renormalization ambiguous part of the distribution function in the slow roll approximation factors out as an inert overall normalization. However, it shows up beyond this approximation and, thus, persists at the fundamental level.

stood. This mode is peculiar because of its ghost nature — its kinetic term enters the action with the wrong sign. This fact was emphasized in [20], but its cosmological consequences have not yet properly been examined. There is a viewpoint that this mode should at all be excluded from the quantum perturbations spectrum on the physical grounds that it is always beyond the horizon and not directly observable [21]. This approach can hardly be justified, because this mode is in some sense the most fundamental one, for it determines the homogeneous background on top of which all observable perturbations are unraveling. The energy of this quantum mechanical mode in view of its ghost nature is not positive definite, and it leads to peculiar back reaction phenomena which we are going to discuss here. Despite their actual small magnitude in our model, in other cases they might essentially contribute to the cosmological evolution. As we shall see, quantum fluctuations of the homogeneous inflaton have the equation of state $p + \varepsilon = 0$ and, thus, it can even be a candidate for the present day cosmological constant, instead of quintessence. This is one more motivation for our studies.

The organization of the paper is as follows. In Section 2 we briefly recapitulate the results of [6,7,18] for the model with a big negative nonminimal coupling of the inflaton, $-\xi R\varphi^2/2$, $-\xi = |\xi| \gg 1$, having a quasi-quartic potential $V(\varphi) = \lambda\varphi^4/4 + \cdots$. We formulate the mechanism of one-loop radiative corrections due to which the inflaton distribution acquires a sharp peak. This peak gives rise to initial conditions for inflation and quantum fluctuations contributing to the radiation currents — quantum terms in the effective equations. In Section 3 we reveal the general structure of these currents and their gauge invariance properties. In particular, their decomposition into the contributions of two sectors is presented: the quantum mechanical sector of the homogeneous inflaton and the field-theoretical sector of inhomogeneous modes. The Euclidean effective action method is formulated for the calculation of the latter, while the former is supposed to be obtained by direct quantum averaging with respect to the peak-like wavefunction of the inflaton. In Section 4 we go over to the generic model with minimally coupled inflaton. For this model we pose the Cauchy problem for the classical solution — the tree-level approximation for expectation values. We perform quantum reduction of the wavefunction from minisuperspace (of the scale factor and inflaton) to the physical subspace, the latter playing the role of the Cauchy surface in minisuperspace at which quantum initial conditions are posed. For the no-boundary and tunneling wavefunctions this subspace is chosen as a domain corresponding to the nucleation of the Lorentzian quasi-de Sitter spacetime from the Euclidean one. In Section 5 a similar quantum Cauchy problem is posed for cosmological perturbations. Their physical reduction is briefly presented along the lines of [20]. Perturbations of minisuperspace variables — scale factor, lapse and inflaton — are expressed in the Newton gauge in terms of the invariant physical variables of [20]. The latter, in their turn, are found as operators in the representation of the physical wavefunction on the Cauchy surface of the above type. In Section 6 we get back to the nonminimal model of Section 2 and discuss its reparametrization to the Einstein frame which allows one to transform its quantum Cauchy problem to that of the minimal model of Sections 4 and 5. We also describe here the calculation of radiation currents and derive the expression for the rolling force in the effective inflaton equation. Sections 7 and 8 present the resulting effects respectively at the beginning of inflation and at late steady inflationary stage. Concluding section contains a brief summary of results and prospective implications of its technique.

2. Quantum origin of the Universe as a low-energy phenomenon

The starting point of our considerations is the assumption that quantum cosmology can give initial conditions for inflation, which in their turn determine main cosmological parameters of the observable Universe, including the density parameter Ω. The requirement of $\Omega > 1$ in closed cosmology gives the bound on the e-folding number N — the logarithmic expansion coefficient for the scale factor a during the inflation stage with a Hubble constant $H = \dot{a}/a$,

$$N = \int_0^{t_F} dt\, H \tag{2.1}$$

(with $t = 0$ and t_F denoting the beginning and the end of inflation epoch). This bound reads $N \geqslant 60$ [12]. On the other hand, the value of N is directly related to the initial conditions for inflation — initial value of the inflaton, φ_I,

$$N \simeq -\int_0^{\varphi_I} d\varphi\, \frac{H(\varphi)}{\dot{\varphi}}. \tag{2.2}$$

In the chaotic inflation model the effective Hubble constant $H = H(\varphi)$ is generated by the inflaton and, therefore, all the parameters of the inflationary epoch can be found as functions of φ_I. This initial condition belongs to the quantum domain, i.e., it is subject to the quantum distribution following from the cosmological wavefunction. If this distribution has a sharp probability peak at certain $\varphi = \varphi_I$, then this value serves as the initial condition for inflation.

There are two known quantum states that lead in the semiclassical regime to the closed inflationary Universe — the no-boundary [2–4] and tunneling [5] wavefunctions. They both describe quantum nucleation of the Lorentzian quasi-de Sitter spacetime from the Euclidean (positive signature) hemisphere — the gravitational instanton responsible for the classically forbidden state of the gravitational field. In the tree-level approximation they generate the distribution functions

$$\rho^{\rm tree}_{\rm NB,T}(\varphi) \sim \exp\bigl[\mp I(\varphi)\bigr], \tag{2.3}$$

where $I(\varphi)$ is the Euclidean action of this instanton with the inflaton φ. For a wide class of monotonically growing potentials with an infinite range of φ (most natural from the viewpoint of quantum field theory of renormalizable matter fields), these functions are unnormalizable in the high-energy domain $\varphi \to \infty$ and generally devoid of the observationally justified probability peaks. However, by including quantum loop effects and applying the theory to a particular model with strong nonminimal curvature coupling of the inflaton one can qualitatively change the situation — generate a sharp probability peak at GUT energy scale satisfying the above bound on N [6,7,18]. The basic formalism underlying this result is as follows.

Beyond the tree level the distribution $\rho_{\mathrm{NB,T}}(\varphi)$ is not just the square of the cosmological wavefunction (2.3). It becomes the diagonal element of the reduced density matrix obtained by tracing out all degrees of freedom but φ. As shown in [18,22–24] in the one-loop approximation it reads

$$\rho_{\mathrm{NB,T}}(\varphi) \sim \exp\bigl[\mp I(\varphi) - \boldsymbol{\Gamma}_{1\text{-loop}}(\varphi)\bigr], \tag{2.4}$$

where the classical action is amended by the Euclidean effective action $\boldsymbol{\Gamma}_{1\text{-loop}}(\varphi)$ of all quantum fields that are integrated out. This action is calculated on the same instanton, and its contribution can qualitatively change predictions of the tree-level theory due to the logarithmic scaling behaviour of $\boldsymbol{\Gamma}(\varphi)$. On the instanton of the size $1/H(\varphi)$ it looks like $\boldsymbol{\Gamma}(\varphi) \sim Z \ln H(\varphi)$, where Z is the total anomalous scaling of all quantum fields in the model.

The model of [6,7] has the graviton–inflaton sector with a big negative constant $-\xi = |\xi| \gg 1$ of nonminimal curvature coupling,

$$S[g_{\mu\nu}, \varphi] = \int d^4x\, g^{1/2} \biggl(\frac{m_P^2}{16\pi} R(g_{\mu\nu}) - \frac{1}{2}\xi\varphi^2 R(g_{\mu\nu}) \\ - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi^4 \biggr), \tag{2.5}$$

and generic GUT sector of Higgs χ, vector gauge A_μ and spinor fields ψ coupled to the inflaton via the interaction term

$$S_{\mathrm{int}} = \int d^4x\, g^{1/2} \biggl(\sum_\chi \frac{\lambda_\chi}{4}\chi^2\varphi^2 + \sum_A \frac{1}{2} g_A^2 A_\mu^2 \varphi^2 \\ + \sum_\psi f_\psi \varphi \bar{\psi}\psi + \text{derivative coupling} \biggr). \tag{2.6}$$

This model is of a particular interest for a number of reasons. Firstly, from the phenomenological viewpoint a strong nonminimal coupling allows one to solve the problem of exceedingly small λ (because here the observable magnitude of CMBR anisotropy $\Delta T/T \sim 10^{-5}$ is proportional to the ratio $\sqrt{\lambda}/|\xi|$ [26,27]). Secondly, this coupling is inevitable from the viewpoint of renormalization theory. Also, among recent implications, it might be important in the theory of an accelerating Universe [25]. Finally, for a wide class of GUT-type particle physics sectors this model generates a sharp probability peak in $\rho_{\mathrm{NB,T}}(\varphi)$ at some $\varphi = \varphi_I$ [6,7]. This peak belongs to the GUT energy scale — the corresponding effective Hubble constant, $H(\varphi_I) = \sqrt{\lambda/12|\xi|}\varphi_I$, is proportional to $m_P\sqrt{\lambda}/|\xi| \sim 10^{-5} m_P$. This, in its turn, justifies the use of GUT for matter part of the model, because this scale is much below the supersymmetry and string theory scales.

The mechanism of this peak is based on a large value of $|\xi|$ and the interaction (2.6) which induces via the Higgs effect large masses of all the particles directly coupled the inflaton. Due to this effect the parameter Z (dominated by terms quartic in particle masses) is quadratic in $|\xi|$, $Z = 6|\xi|^2 \boldsymbol{A}/\lambda$, with a universal combination of the coupling constants from (2.6)

$$\boldsymbol{A} = \frac{1}{2\lambda}\left(\sum_\chi \lambda_\chi^2 + 16\sum_A g_A^4 - 16\sum_\psi f_\psi^4\right). \tag{2.7}$$

Therefore, the probability peak in this model reduces to the extremum of the function

$$\ln \rho_{\mathrm{NB,T}}(\varphi) \simeq \mathrm{const} \pm \frac{24\pi(1+\delta)|\xi|}{\lambda}\frac{m_P^2}{\varphi^2} - 3\frac{|\xi|^2}{\lambda}\boldsymbol{A}\ln\frac{\varphi^2}{\mu^2} + \mathrm{O}\left(\frac{m_P^4}{\varphi^4}\right). \tag{2.8}$$

Here the φ-dependent part of the classical instanton action is taken in the lowest order of the slow roll smallness parameter, $m_P^2/|\xi|\varphi^2 \ll 1$, renormalization ambiguous parameter μ enters only the overall normalization of $\rho_{\mathrm{NB,T}}(\varphi)$ and

$$\delta \equiv -\frac{8\pi|\xi|m^2}{\lambda m_P^2}. \tag{2.9}$$

For the no-boundary and tunneling states the peak exists for positive $\boldsymbol{A}$ and respectively negative and positive values of $1+\delta$, $\pm(1+\delta) < 0$. The parameters of this peak — mean values of the inflaton and Hubble constants and the quantum width $\varDelta$,

$$\varphi_I = m_P\sqrt{\frac{8\pi|1+\delta|}{|\xi|\boldsymbol{A}}}, \qquad H(\varphi_I) = m_P\frac{\sqrt{\lambda}}{|\xi|}\sqrt{\frac{2\pi|1+\delta|}{3\boldsymbol{A}}}, \tag{2.10}$$

$$\varDelta = \frac{1}{\sqrt{12\boldsymbol{A}}}\frac{\sqrt{\lambda}}{|\xi|}\varphi_I, \tag{2.11}$$

are strongly suppressed by a small ratio $\sqrt{\lambda}/|\xi|$ known from the COBE normalization for $\Delta T/T \sim 10^{-5}$ [8,9]. Because of small width the distribution function can be approximated by the gaussian packet

$$\rho_{\mathrm{NB,T}}(\varphi) \simeq \frac{1}{\varDelta\sqrt{2\pi}}\exp\left[-\frac{(\varphi-\varphi_I)^2}{2\varDelta^2}\right]. \tag{2.12}$$

It is important that the initial conditions (2.10), (2.11) are well out of range of eternal inflation and self-reproduction. The expression for self-reproduction scale is well known for the model with minimally coupled field for a wide class of slow-roll potentials. But the model (2.5) can be transformed to the Einstein frame of new metric $\bar{g}_{\mu\nu}$ and inflaton $\bar{\varphi}$ fields minimally coupled to one another. This transformation and, in particular, the potential of the inflaton in the minimal frame $\overline{V}(\bar{\varphi})$ are given in Section 6 below. The condition for eternal inflation, $\overline{H}^2/2\pi\dot{\bar{\varphi}} > 1$, written in this frame in terms of $\overline{V}(\bar{\varphi})$,

$$m_P^3\frac{d\overline{V}/d\bar{\varphi}}{\overline{V}^{3/2}}\bigg|_{\bar{\varphi}_*} \sim 1, \tag{2.13}$$

gives the value $\bar{\varphi}_*$ beyond which eternal inflation and self-reproduction begin. Using Eqs. (6.9), (6.10) and transforming the result back to the original nonminimal frame, one gets the critical value

$$\varphi_* \simeq m_P \left(\frac{2}{\lambda}\right)^{1/4} (1+\delta)^{1/2} \gg \varphi_I, \tag{2.14}$$

which turns out to be far away from the probability peak — by the order of magnitude $(|\xi|/\sqrt{\lambda})^{1/2}$ times bigger than φ_I. It is instructive to calculate the parameters of the model at φ_*. The corresponding Hubble constant and the inflaton potential read

$$H(\varphi_*) \simeq m_P \left(\frac{\sqrt{\lambda}}{|\xi|}\right)^{1/2} (1+\delta)^{1/2} \gg H(\varphi_I), \tag{2.15}$$

$$V(\varphi_*) \simeq m_P^4 (1+\delta)^2. \tag{2.16}$$

Thus, the spacetime curvature at the self-reproduction threshold is much bigger than in our initial conditions, but still remains below the Planck scale. On the contrary, the inflaton energy density reaches Planckian value and, therefore, enters non-perturbative domain. [4] This brings certain doubt on the conventional semiclassical mechanism of eternal inflation in this model. This is one more reason to believe that eternal inflation does not wipe out quantum initial conditions.

The value of the parameter (2.9) is crucial for the inflationary evolution from this gaussian peak (2.12). The classical equations of motion in the slow roll approximation,

$$\ddot{\varphi} + 3H\dot{\varphi} - F(\varphi) = 0, \tag{2.17}$$

$$H(\varphi) \simeq \sqrt{\frac{\lambda}{12|\xi|}}\,\varphi, \qquad F(\varphi) \simeq -\frac{\lambda m_P^2 (1+\delta)}{48\pi\xi^2}\varphi, \tag{2.18}$$

show that the inflaton decreases from its initial value, $\dot{\varphi} \simeq F/H < 0$, only for $1+\delta > 0$, that is only for the *tunneling* state (minus sign in (2.8)). Only in this case the duration of the inflationary epoch is finite with the e-folding number (2.2) [1]

$$N \simeq \frac{6\pi|\xi|\varphi_I^2}{m_P^2(1+\delta)} = \frac{48\pi^2}{\boldsymbol{A}}. \tag{2.19}$$

Comparison with $N \geqslant 60$ immediately yields the bound $\boldsymbol{A} \leqslant 5.5$ which can be regarded as a selection criterion for particle physics models [6]. This conclusion remains qualitatively true when taking into account the contribution of the inhomogeneous quantum modes to the radiation current of the effective equations [1]. This contribution and its dynamical effect were obtained in [1] by the method of the Euclidean effective action, however, the quantum fluctuations of the inflaton field itself have not been taken into account.

For the proponents of the no-boundary quantum states in a long debate on the wavefunction discord [28–31] this situation might seem unacceptable. According to this result the no-boundary proposal does not generate realistic inflationary scenario, while the tunneling state does not satisfy important aesthetic criterion — the universal formulation of the initial conditions and dynamical aspects in one concept — spacetime covariant

[4] This mismatch between curvature and energy density is explained by the fact that the effective Planck mass in nonminimally coupled model is much bigger than the bare one, see Eq. (6.6) below.

path integral over geometries,[5] not to say about intrinsic inconsistency mentioned in Introduction. Thus, one of the motivations of considering the quantum mechanical sector of the inflaton mode is the hope that it can handle this difficulty. In view of the smallness of the quantum width (2.11) the quantum fluctuations $\Delta\varphi \sim \varDelta$ are expected to be negligible, but those of their quantum momenta $\Delta p_\varphi \sim 1/\varDelta$ blow up for small $\varDelta$. Therefore, a priori, it is hard to predict the overall magnitude of the quantum rolling force and its sign due to $\Delta\varphi(t)$. In what follows we carefully consider this problem.

3. Effective equations: setting the problem

Effective equations for expectation values of operators in the quantum state $|\boldsymbol{\Psi}\rangle$,

$$g(x) = \langle\boldsymbol{\Psi}|\hat{g}(x)|\boldsymbol{\Psi}\rangle, \tag{3.1}$$

$$\hat{g}(x) = \hat{\varphi}(x), \hat{\chi}(x), \hat{\psi}(x), \hat{A}_\mu(x), \hat{g}_{\mu\nu}(x), \ldots, \tag{3.2}$$

can be obtained by expanding the Heisenberg equations of motion, $\delta S[\hat{g}]/\delta\hat{g}(x) = 0$, for the full quantum field $\hat{g}(x) = g(x) + \Delta\hat{g}(x)$ in powers of quantum disturbances $\Delta\hat{g}(x)$

$$\frac{\delta S[g]}{\delta g(x)} + \int dy\, \frac{\delta^2 S[g]}{\delta g(x)\delta g(y)} \Delta\hat{g}(y) + \frac{1}{2}\int dy\, dz\, \frac{\delta^3 S[g]}{\delta g(x)\delta g(y)\delta g(z)} \Delta\hat{g}(y)\Delta\hat{g}(z) + \cdots = 0, \tag{3.3}$$

and averaging them with respect to $|\boldsymbol{\Psi}\rangle$. The linear in $\Delta\hat{g}(x)$ term identically drops out of this expansion, because $\langle\Delta\hat{g}(x)\rangle \equiv 0$, and the effective equations acquire a generic form

$$\frac{\delta S[g]}{\delta g(x)} + J(x) = 0. \tag{3.4}$$

Here $S[g]$ is the classical action of the system, and the radiation current $J(x)$ accumulates all quantum corrections which begin with the one-loop contribution

$$J(x) = \frac{1}{2}\int dy\, dz\, \frac{\delta^3 S[g]}{\delta g(x)\delta g(y)\delta g(z)} G(z, y) + \cdots. \tag{3.5}$$

The Wightman function of quantum disturbances $G(z, y)$ in a given quantum state $|\boldsymbol{\Psi}\rangle$

$$G(z, y) = \langle\boldsymbol{\Psi}|\Delta\hat{g}(z)\Delta\hat{g}(y)|\boldsymbol{\Psi}\rangle, \tag{3.6}$$

$$\Delta\hat{g}(y) \equiv \hat{g}(y) - g(y), \tag{3.7}$$

can be found by iterations as a loop expansion in powers of $\hbar$. Because semiclassically $\Delta\hat{g} = \mathrm{O}(\hbar^{1/2})$ and $J(x) = \mathrm{O}(\hbar)$, it follows from Eqs (3.3) and (3.4) that the linear in $\Delta\hat{g}$ term of (3.3) is at least linear in $\hbar$. Therefore, in the one-loop approximation the quantum perturbation $\Delta\hat{g}(y)$ can be identified with the solution of the linearized classical equation

[5] The Lorentzian path integral for the tunneling state of [30] also requires, in this respect, extension beyond minisuperspace level, development of the semiclassical expansion technique, etc.

on the mean-field background

$$\int dy \, \frac{\delta^2 S[g]}{\delta g(x)\delta g(y)} \Delta\hat{g}(y) = 0. \tag{3.8}$$

Its solution can be parametrized by the operator-valued initial conditions. Depending on the representation of the initial state $|\boldsymbol{\Psi}\rangle$, they can be either the creation–annihilation operators, or operators of initial fields and their conjugated momenta, so that quantum averaging in (3.6) becomes straightforward. Continuing this procedure by iterations one can obtain the radiation current in any loop order as a complicated but, in principle, calculable functional of the mean field.

Alternatively to (3.5), (3.6), the one-loop radiation current can be written as

$$J^{1\text{-loop}}(x) = \frac{1}{2}\left\langle \left[\frac{\delta S[\hat{g}]}{\delta\hat{g}(x)}\right]_2 \right\rangle, \tag{3.9}$$

where $[\ldots]_2$ denotes the quadratic part of the quantity expanded in powers of disturbances $\Delta\hat{g}$ that solve linearized classical equations, and $\langle\ldots\rangle$ implies the quantum averaging with respect to $|\boldsymbol{\Psi}\rangle$.

For the cosmological system this simple perturbative scheme should, however, be amended by two important aspects. One of them reflects the local gauge (general coordinate) invariance of the problem and the other deals with disentangling the collective variables. The latter describes the most important (minisuperspace) cosmological degrees of freedom having non-trivial expectation values. In the next two sections we consider these two aspects of the problem.

3.1. Gauge properties of the radiation current

In view of local general coordinate and other gauge invariances, fields and their perturbations contain purely gauge variables that should be gauged away. Thus, the physical sector should be disentangled from the full configuration space of the system and the physical state should be prescribed on this physical sector. This is the general scheme of the reduced phase space quantization [24,32,33]. For describing this scheme in application to perturbative radiation currents we simplify the formalism by using condensed notations for the full set of fields (3.1), $g^a = g(x)$, in which the condensed index a includes both discrete spin labels and spacetime coordinates x. Contraction of these indices implies also the spacetime integration. In these notations, the invariance of the action $S[g]$ under local gauge transformations, $g^a \to g^a + R^a_\mu f^\mu$, with infinitesimal gauge parameters f^μ (the condensed index μ bearing together with discrete tensor labels spacetime arguments of the local function $f^\mu = f(x)$) reads

$$R^a_\mu \frac{\delta S}{\delta g^a} = 0. \tag{3.10}$$

Here, R^a_μ is a generator of the gauge transformation — the quasilocal two-point kernel with respect to spacetime coordinates associated with condensed labels a and μ.

A gauge breaking procedure — a part of the physical reduction — can be enforced by adding to the classical action the gauge-breaking term and, in the quantum domain —

for Heisenberg equations — by including the action of Faddeev–Popov ghosts. Then the derivation of effective equations repeats the perturbative scheme of the above type with gauge-breaking and ghost terms included into the full action. However, for the purpose of physical reduction gauge conditions should be unitary, that is transforming under gauge transformations locally in time (not involving time derivatives of the gauge parameters f^μ). In such a gauge the ghosts are not propagating, and the ghost action can be omitted from the total quantum system. As a result, the one-loop effective equations again take the form (3.4) with the same radiation current

$$J_a^{\text{1-loop}} = \frac{1}{2}\frac{\delta^3 S}{\delta g^a \delta g^b \delta g^c}\langle \Delta\hat{g}^b \Delta\hat{g}^c\rangle. \tag{3.11}$$

The only modification due to local invariance is that the linearized equations of motion (3.8) for quantum perturbations $\Delta\hat{g}^a$ are supplied with the linear gauge conditions

$$\frac{\delta^2 S}{\delta g^a \delta g^b}\Delta\hat{g}^b = 0, \tag{3.12}$$

$$\chi_a^\mu \Delta\hat{g}^a = 0. \tag{3.13}$$

The functional matrix (two-point kernel) of the linear gauge condition χ_a^μ is assumed to form the Faddeev–Popov operator

$$Q_\nu^\mu \equiv \chi_a^\mu R_\nu^a, \tag{3.14}$$

which is ultralocal in time, $Q_\nu^\mu \sim \delta(t_\mu - t_\nu)$ (the property of the unitary gauge), and invertible. In view of this ultralocality the inverse of Q_ν^μ, $Q_\mu^{-1\nu}$, does not require imposing any boundary conditions in the past or future of the time variable.

The system of Eqs. (3.12), (3.13) for quantum perturbations $\Delta\hat{g}^b$ has a number of peculiarities. First, the linearized gauge condition (3.13) guarantees that the gauge-breaking term (usually quadratic in gauge conditions) does not contribute to the radiation current (3.11). Second, it fixes uniquely the solution for $\Delta\hat{g}^a$ under given initial conditions. In the absence of gauge conditions, Eq. (3.12) would have the ambiguity in the solution, $\Delta\hat{g}^a \to \Delta\hat{g}^a + R_\mu^a \hat{f}^\mu$, with arbitrary $\hat{f}^\mu$ because of a simple corollary of (3.10)

$$\frac{\delta^2 S}{\delta g^a \delta g^b} R_\mu^a = -\frac{\delta S}{\delta g^a}\frac{\delta R_\mu^a}{\delta g^b}. \tag{3.15}$$

Here the right-hand side vanishes on the classical solution, $\delta S/\delta g^a = 0$. So, the gauge generators R_μ^a are zero vectors of the Hessian of the action on this background, which implies the gauge invariance of the linearized solution of the above type. However, the auxiliary condition (3.13) gauges this invariance away and guarantees the uniqueness of the solution for $\Delta\hat{g}^a$.

The parametrization of the general solution of Eqs. (3.12), (3.13) in terms of the symplectic (phase space) initial conditions is equivalent to the Hamiltonian reduction of this linearized system to the physical sector. This reduction should be done in the canonical formalism. The unitarity of gauge conditions (3.13) guarantees that they can be rewritten in terms of the phase space variables — configuration coordinates and conjugated momenta

— contained in the set of $\Delta\hat{g}^a$ and $d\Delta\hat{g}^a/dt$ (the rest of the variables in $\Delta\hat{g}^a$ are the Lagrange multipliers). Solving these canonical gauge conditions together with the first class constraints — the nondynamical subset of Eq. (3.12) — one finds all the perturbations $\Delta\hat{g}^a$ as functions of the physical variables $\Delta\hat{g}_{\rm phys}$ which in their turn become functions of initial physical coordinates and momenta $(\boldsymbol{q}_0, \boldsymbol{p}_0)$

$$\Delta\hat{g}^a = \Delta\hat{g}^a(\Delta\hat{g}_{\rm phys}), \tag{3.16}$$

$$\Delta\hat{g}_{\rm phys} = \Delta\hat{g}_{\rm phys}(\boldsymbol{q}_0, \boldsymbol{p}_0). \tag{3.17}$$

If the initial quantum state is known on the physical sector in the representation of these quantum variables, $|\boldsymbol{\Psi}\rangle = \boldsymbol{\Psi}(\boldsymbol{q}_0)$, $\hat{\boldsymbol{p}}_0 = \partial/i\partial\boldsymbol{q}_0$, then the calculation of averages in (3.11) becomes straightforward.

In what follows we use this calculational strategy. The no-boundary and tunneling wavefunctions as solutions of the Wheeler–DeWitt equations on superspace will, first, be reduced to the physical sector. This quantum reduction in the one-loop approximation will be performed by the technique of [24,32,33].[6] Simultaneously, the physical reduction will be performed for the cosmological background and its perturbations, the both being parametrized in terms of initial data encoded in the (reduced) physical wavefunction. This makes further calculation of radiation currents straightforward.

We finish this section with gauge invariance of the radiation current — the corollary of the Noether identity for the classical action (3.10). The latter, after two consequitive functional differentiations, yields another identity

$$R^a_\mu \frac{\delta^3 S}{\delta g^a \delta g^b \delta g^c} = -\frac{\delta R^a_\mu}{\delta g^b}\frac{\delta^2 S}{\delta g^a \delta g^c} - \frac{\delta R^a_\mu}{\delta g^c}\frac{\delta^2 S}{\delta g^a \delta g^b} - \frac{\delta^2 R^a_\mu}{\delta g^b \delta g^c}\frac{\delta S}{\delta g^a}. \tag{3.18}$$

Using it, one shows on account of the linearized equations (3.12) that the radiation current satisfies the relation

$$R^a_\mu J^{\text{1-loop}}_a = -\frac{\delta^2 R^a_\mu}{\delta g^b \delta g^c}\frac{\delta S}{\delta g^a}\langle\Delta\hat{g}^b\Delta\hat{g}^c\rangle. \tag{3.19}$$

Here the right-hand side vanishes on the classical background, $\delta S/\delta g^a = 0$, and, moreover, on an *arbitrary* mean field background when the generator is linear in the field, $\delta^2 R^a_\mu/\delta g^b \delta g^c = 0$. But this is a well known property of the generators of general coordinate transformations that form the closed algebra of spacetime diffeomorphisms. Thus, the one-loop radiation currents also satisfy the Noether identity

$$R^a_\mu J^{\text{1-loop}}_a = 0. \tag{3.20}$$

This property will be very important in what follows. It implies that the radiation currents are linearly dependent, which reduces the number of effective equations to be solved

[6] The fact that the cosmological states are known as solutions of quantum Dirac constraints on superspace, and the fact that their quantum reduction to the physical sector is readily available from [24,32,33], explains why we work within unitary gauge fixing procedure. Relativistic gauges with propagating Faddeev–Popov ghosts would require a quantum state on extended Hilbert space with indefinite metric, satisfying the zero BRST-charge equation (see review of this problem in [24]). Lifting the Dirac wavefunctions of the no-boundary and tunneling states to this space, to the best of our knowledge, has not been done and goes beyond the scope of this paper.

in cosmological applications. Moreover, this identity reflects the gauge invariance of effective equations themselves. In particular, for purely gravitational system with $g^a(x) = g_{\mu\nu}(x)$, when the radiation current coincides with the expectation value of the stress tensor $J^{\mu\nu\text{1-loop}}(x) = \langle\hat{T}^{\mu\nu}(x)\rangle$, this property signifies the covariant conservation law, $\nabla_\mu\langle\hat{T}^{\mu\nu}(x)\rangle = 0$.

3.2. Two configuration space sectors of the model

In closed cosmological model, the total metric and inflaton scalar field are usually decomposed into the spatially homogeneous background and inhomogeneous perturbations

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\gamma_{ij}\,dx^i\,dx^j + h_{\mu\nu}(x)\,dx^\mu\,dx^\nu, \tag{3.21}$$

$$\varphi(x) = \varphi(t) + \delta\varphi(x), \quad x = (t, \mathbf{x}), \tag{3.22}$$

where $a(t)$ is the scale factor, $N(t)$ is the lapse function and γ_{ij} is the spatial metric of the 3-sphere of unit radius. Therefore, the full set of fields consists of the minisuperspace sector of spatially homogeneous variables $Q(t)$ and inhomogeneous fields $f(x)$ essentially depending on spatial coordinates $x^i = \mathbf{x}$

$$g(x) = Q(t), f(t, \mathbf{x}), \tag{3.23}$$

$$Q(t) = a(t), \varphi(t), N(t), \tag{3.24}$$

$$f(x) = \delta\varphi(t, \mathbf{x}), h_{\mu\nu}(t, \mathbf{x}), \chi(t, \mathbf{x}), \psi(t, \mathbf{x}), A_\mu(t, \mathbf{x}), \ldots. \tag{3.25}$$

From the structure of the initial quantum state, that will be discussed later, it follows that only minisuperspace variables have nonvanishing expectation values

$$\langle\widehat{Q}(t)\rangle \neq 0, \qquad \langle\hat{f}(x)\rangle = 0. \tag{3.26}$$

Therefore, the full set of effective equations reduces to the following three equations in the minisuperspace sector

$$\begin{aligned}&\frac{\delta S[Q]}{\delta Q(t)} + J_Q(t) = 0,\\ &J_Q = J_N, J_a, J_\varphi,\end{aligned} \tag{3.27}$$

their quantum radiation currents $J_Q(t)$ containing the contribution of quantum fluctuations of minisuperspace modes themselves and those of spatially inhomogeneous fields.

The set of these equations is, however, redundant in view of the Noether identities for both classical (3.10) and quantum (3.20) parts. In the minisuperspace sector the general coordinate transformations reduce to reparametrizations of time. Infinitesimal transformations of minisuperspace variables $Q(t) = (N(t), a(t), \varphi(t))$, in the notations of Section 3.1, read as: $f^\mu \equiv f(t)$, $\nabla^a_\mu f^\mu \equiv \nabla^Q f = (d(Nf)/dt, \dot{a}f, \dot{\varphi}f)$, and the identity (3.20) takes the form

$$\dot{\varphi}J_\varphi + \dot{a}J_a - N\dot{J}_N = 0. \tag{3.28}$$

The currents J_N and J_φ have direct physical interpretation in terms of the quantum energy density ε and pressure p

$$J_N = -a^3\sqrt{\gamma}\,\varepsilon, \qquad J_a = 3a^2\sqrt{\gamma}\,p, \tag{3.29}$$

so that Eq. (3.28) in the cosmic time, $N = 1$, takes the form

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p) + \frac{J_\varphi}{\sqrt{\gamma}a^3}\dot{\varphi} = 0. \tag{3.30}$$

As we shall see later, the third current J_φ can be interpreted in terms of the quantum rolling force driving the evolution of the inflaton field. Therefore, this equation measures the balance of the conservation for the quantum stress tensor vs the work of this force. In the slow roll regime, $\dot{\varphi} \simeq 0$, it reduces to the conventional covariant conservation law on the Robertson–Walker background.

The Hamiltonian reduction to the physical sector, discussed above,

$$\Delta g(x) = \big(\Delta Q(t), f(x)\big) \to \Delta g_{\rm phys}(x) = \big(\Delta Q_{\rm phys}(t), f_{\rm phys}(x)\big), \tag{3.31}$$

$$f_{\rm phys}(x) = \big(h^{\rm TT}(x), \text{matter fields}\big), \tag{3.32}$$

leaves us with the set of physical variables $\Delta g_{\rm phys}(x)$ which also splits into minisuperspace and field-theoretical subsets. Here $\Delta Q_{\rm phys}(t)$ is a single field variable that originates from the minisuperspace sector of metric and inflaton perturbations.[7] The rest, $f_{\rm phys}(x)$, represent the transverse traceless modes of the gravitational wave $h^{\rm TT}(x)$ and other physical nongravitational degrees of freedom. The nature of $\Delta Q_{\rm phys}(t)$ depends on the gauge used for disentangling the physical sector. A particular gauge fixing procedure widely used in the theory of cosmological perturbations [20,34] picks up a special gauge invariant variable $\Delta Q_{\rm phys}(t) = \boldsymbol{q}(t)$ that will be discussed in much detail in Section 5. At the nonlinear level (beyond perturbation theory on a classical background) another choice of $Q_{\rm phys}(t)$ is possible by simply identifying it with the spatially homogeneous inflaton field $\varphi(t)$, $Q_{\rm phys}(t) = \varphi(t)$. In both cases, particular expressions for the original minisuperspace variables $Q(t)$ and their perturbations $\Delta Q(t)$ in terms of $Q_{\rm phys}(t)$ and $\Delta Q_{\rm phys}(t)$ depend on the choice of gauge conditions. Two types of these gauge conditions will be considered in Sections 4 and 5.

Splitting the whole configuration space into minisuperspace and inhomogeneous sectors (3.23) reflects the choice of the collective variables. Moreover, it reflects distinctly different nature of quantum states for the modes belonging to these two sectors. This results in different calculational strategies for the corresponding quantum averages. To see it, note that on the space of physical variables $(\varphi, f_{\rm phys}(\mathbf{x}))$, with $f_{\rm phys}(\mathbf{x})$ treated perturbatively, the initial no-boundary and tunneling wavefunctions read (see Section 4)

$$\boldsymbol{\Psi}^{\text{1-loop}}(\varphi, f_{\rm phys}) = P(\varphi)\exp\left[\mp\frac{1}{2}I(\varphi) - \frac{1}{2}\Omega(\varphi)f_{\rm phys}^2 + O\big(f_{\rm phys}^3\big)\right]. \tag{3.33}$$

[7] Counting the number of physical degrees of freedom is usual: in 3-dimensional configuration space of (N, a, φ) subject to one first class constraint the number of physical degrees of freedom equals $3 - 2 \times 1 = 1$.

Here $P(\varphi)$ is the loop prefactor and the tree-level exponential contains the Euclidean action expanded up to a quadratic term in inhomogeneous modes, $\Omega(\varphi) f_{\rm phys}^2$.

From (3.33) it follows that the one-loop quantum correlators between the minisuperspace modes and inhomogeneous fields vanish, [8] therefore they contribute additively to the total one-loop radiation current

$$J = J^q + J^f, \tag{3.34}$$

$$J^q(t) = \frac{1}{2}\int dt'\,dt''\,\frac{\delta^3 S[Q]}{\delta Q(t)\delta Q(t')\delta Q(t'')}\big\langle \Delta\widehat{Q}(t')\Delta\widehat{Q}(t'')\big\rangle, \tag{3.35}$$

$$J^f(t) = \frac{1}{2}\int dx\,dy\,\frac{\delta^3 S[Q+f]}{\delta Q(t)\delta f(x)\delta f(y)}\bigg|_{f=0}\big\langle \hat f(x)\hat f(y)\big\rangle. \tag{3.36}$$

Similarly to (3.9) these contributions represent the quantum averages of the quadratic terms of the expansion of $\delta S/\delta Q$ correspondingly in $\Delta\widehat{Q}(t)$ and $\hat f(x)$. However, the calculation methods for J^q and J^f are very different, and the difference can be attributed to qualitatively different quantum states of the modes φ and $f_{\rm phys}$. Let us begin with the radiation current J^f which can be obtained by the effective action method [1].

The matrix of quantum dispersions $\Omega(\varphi)$ in the gaussian state of inhomogeneous modes is such that this state turns out to be the Euclidean quasi-de Sitter invariant vacuum [17,35]

$$|{\rm vac}\rangle_{\rm DS} = C(\varphi)\exp\left[-\frac{1}{2}\Omega(\varphi) f_{\rm phys}^2\right]. \tag{3.37}$$

The corresponding quantum averages

$${}_{\rm DS}\langle {\rm vac}|\, f_{\rm phys}(x) f_{\rm phys}(y)|{\rm vac}\rangle_{\rm DS} = G_{\rm DS}(x,y) \tag{3.38}$$

are given by de Sitter invariant Green's functions which can be obtained by analytic continuation from the unique Green's function on the Euclidean section of the de Sitter spacetime — the inverse of the Hessian of the Euclidean action

$$G_{\rm DS}(x,y) = G_E(x_E,y_E)\big|_{++++\to-+++}, \tag{3.39}$$

$$\int dy_E\,\frac{\delta^2 \boldsymbol{I}[Q+f]}{\delta f(x_E)\delta f(y_E)}\bigg|_{f=0} G_E(y_E,z_E) = \delta(x_E,z_E), \tag{3.40}$$

$$\boldsymbol{I}[Q+f] = -iS[Q+f]\big|_{++++\to-+++}. \tag{3.41}$$

Here x_E denotes the coordinates on the Euclidean de Sitter manifold related by the analytic continuation to the Lorentzian spacetime coordinates, $x_E^4 = \pi/2H + ix^0$, $\mathbf{x}_E = \mathbf{x}$, with H — the Hubble constant or the inverse radius of the Euclidean 4-sphere [9] and $x^0 = t$ —

[8] Note that due to the gaussian nature of the state $f \sim \hbar^{1/2}$, so that the terms contributing to $\langle f\Delta\varphi\rangle$-correlators, $O(f^3) \sim \hbar^{3/2}$, go beyond the one-loop approximation.

[9] The Lorentzian Green's function $G_{\rm DS}(x,y) = iG_{\rm DS}^{(+)}(x,y)$ is the positive frequency Wightman function — the solution of the homogeneous linearized equation of motion. On the contrary, $G_E(x_E,y_E)$, as an inverse of the Hessian, solves the inhomogeneous equation. However, the Wightman function can be obtained from $G_E(x_E,y_E)$ by taking the boundary value of its analytic continuation on a proper shore of the cut in the complex plane of $[\sigma(x,y)]^{1/2}$ — the geodetic distance between the points x and y [17].

the cosmic time in the unperturbed de Sitter metric (3.21) corresponding to $N = 1$, $a(t) = \cosh(Ht)/H$.

In view of this relation and in view of a similar analytic continuation rule between the Lorentzian, $\delta^3 S/\delta Q\delta f^2$, and Euclidean, $\delta^3 \boldsymbol{I}/\delta Q\delta f^2$, 3-vertices, one finds the *Euclidean effective action* algorithm for the radiation current of inhomogeneous quantum modes

$$J^f\big(x^0\big) = -\frac{\delta \boldsymbol{\Gamma}_{\text{1-loop}}[Q]}{\delta Q(x^4)}\bigg|_{++++\to-+++}, \tag{3.42}$$

$$\boldsymbol{\Gamma}_{\text{1-loop}}[Q] = \frac{1}{2}\,\mathrm{Tr}\ln \frac{\delta^2 \boldsymbol{I}[Q+f]}{\delta f(x_E)\delta f(y_E)}\bigg|_{f=0}. \tag{3.43}$$

Note that this is exactly the functional that yields the one-loop contribution to the distribution function (2.4), when evaluated at the classical solution for the minisuperspace background $Q(x^4) = Q(x^4, \varphi)$ parametrically depending on φ,

$$\boldsymbol{\Gamma}_{\text{1-loop}}(\varphi) = \boldsymbol{\Gamma}_{\text{1-loop}}\big[Q\big(x^4, \varphi\big)\big]. \tag{3.44}$$

This algorithm was used in the previous paper [1] for the calculation one-loop radiation currents of f-modes. In [1] $\boldsymbol{\Gamma}_{\text{1-loop}}[Q]$ was obtained by the local Schwinger–DeWitt expansion — the expansion in spacetime derivatives of the background fields, which in the cosmological context corresponds to the slow-roll expansion. [10] Within this expansion the one-loop action is represented as a spacetime integral of the effective Lagrangian expanded in powers of curvatures, matter field strengths and their covariant derivatives. Therefore, the analytic continuation rule (3.42) is trivial — the current $J^f(x^0)$ can be given by the functional variation of the *local Lorentzian one-loop action*

$$J^f\big(x^0\big) = \frac{\delta S_{\text{1-loop}}[Q]}{\delta Q(x^0)}, \tag{3.45}$$

$$S_{\text{1-loop}}\big[Q\big(x^0\big)\big] = i\boldsymbol{\Gamma}_{\text{1-loop}}\big[Q\big(x^4\big)\big]\big|_{++++\to-+++}. \tag{3.46}$$

Therefore, within the local expansion the J^f part of the radiation current can be absorbed in the functional variation of the total *Lorentzian* effective action $S_{\text{eff}}[Q]$, and the effective equations acquire the final form

$$\frac{\delta S_{\text{eff}}[Q]}{\delta Q(x^0)} + J^q\big(x^0\big) = 0, \tag{3.47}$$

$$S_{\text{eff}}[Q] = S[Q] + S_{\text{1-loop}}[Q]. \tag{3.48}$$

Here $S_{\text{eff}}[Q]$ can be obtained from the classical action $S[Q]$ by adding loop corrections to the classical coefficient functions in the curvature and gradient expansion of the

[10] The algorithm (3.42), (3.43) looks as a generalization of the analytic continuation method for the effective equations in asymptotically flat spacetime [38]. Strictly speaking, this algorithm as derived above holds only for *exact* de Sitter background, while the method of [38] was proven for arbitrary asymptotically flat backgrounds that are perturbatively related to flat spacetime. For the deviations from the de Sitter geometry (measured by the magnitude of the slow-roll smallness parameter) the relations (3.42), (3.43) hold for local terms of the effective action, and might not be true for an essentially nonlocal part that cannot be expanded in powers of derivatives. But for the slow-roll inflation regime, which we use throughout the paper, such an expansion — the local Schwinger–DeWitt series — is definitely applicable, which justifies the effective action method.

Lagrangian. [11] These corrections, in their turn, can be calculated in the Euclidean spacetime by the local Schwinger–DeWitt technique [39]. For our model these corrections has been obtained in [1]. Thus, this essential simplification in treating the J^f-part of the radiation current exists due to the two important aspects of the problem — de Sitter-invariant vacuum of f-modes and the slow roll approximation. In contrast to this, to the best of our knowledge, no simplification is available in the calculation of the quantum mechanical part of the current J^q.

To begin with, the wavefunction of the quantum mechanical minisuperspace mode φ is not gaussian. Moreover, in the tree-level approximation the graph of the probability distribution (2.3) is very flat. It does not have good probability peaks and is even unnormalizable. This means that the tree-level quantum averages $\langle\Delta Q\Delta Q\rangle^{\rm tree}$ are badly defined. Beyond the tree-level approximation the situation can be improved, because they should now be defined with the aid of the reduced density matrix

$$\big\langle\Delta Q(t)\Delta Q(t')\big\rangle = \mathrm{tr}\big[\Delta\widehat{Q}(t)\Delta\widehat{Q}(t')\hat\rho\big], \tag{3.49}$$

$$\hat\rho \equiv \rho(\varphi,\varphi') = \int df\,\boldsymbol{\Psi}(\varphi,f)\boldsymbol{\Psi}^*(\varphi',f), \tag{3.50}$$

which originates from tracing the f-variables out and includes loop corrections. As shown in [22–24,41], the diagonal element of this density matrix — the distribution function of φ,

$$\rho(\varphi) = \rho(\varphi,\varphi), \tag{3.51}$$

is given in the approximation of a gaussian integral by the effective action algorithm (2.4). Effective action contributes the factor that can generate a sharp probability peak (2.12) with the dispersion $\varDelta$ defined by (2.11). With this modification the quantum correlators become well defined, being expressed in terms of $\langle\Delta\varphi\Delta\varphi\rangle \sim \varDelta^2 < \infty$. Certainly, this improvement is achieved by exceeding the precision of the one-loop approximation — badly defined tree-level quantum correlators become finite due to one-loop contribution (therefore, in their turn they effectively contribute to the radiation currents two-loop quantities). But overstepping the conventional rules of the loop expansion is justified here because it reflects the underlying physics of the slow roll dynamics.

Point is that the inflaton field in models satisfying the slow-roll conditions effectively represents the massless scalar field — its mass is roughly proportional to the slow-roll smallness parameter [7]. But massless scalar fields do not have a well defined de Sitter invariant vacuum [17]. This fact, in particular, manifests itself in the unnormalizability of the tree-level wavefunction $\exp[\mp I(\varphi)/2]$, absence of its local maxima, etc. As we see, loop effects render this state a quasi-gaussian nature (2.12) and thus justify the improved

[11] Note that the notion of $S_{\rm eff}[Q]$, as a generator of equations for expectation values, is legitimate only within the local derivative expansion. For nonlocal contributions this action does not exist at all — there is no mean field functional that could yield by the variational procedure effective equations for expectation values [38]. The reason of this is that for nonlocalities the analytic continuation from the Euclidean to Lorentzian spacetime is not unique — in addition, it requires setting the retardation boundary conditions for nonlocal form factors (see [38] and cf. the previous footnote). These boundary conditions prohibit the existence of the effective action for expectation values.

semiclassical expansion. A major part of the paper in what follows deals with the direct calculation of the quantum mechanical radiation current J^q and its physical implications.

4. Quantum Cauchy problem: tree level approximation

Loop expansion for effective equations is essentially perturbative. Therefore, we solve them by iterations starting with the classical solution. Then, in the one-loop approximation the radiation current can be calculated on the classical background — the lowest order approximation for the mean field. Here we pose the initial conditions for this solution that follow from the no-boundary and tunneling cosmological wavefunctions.

In this and the next section we work with the model of minimally coupled inflaton field ϕ having a generic potential $V(\phi)$ (we reserve the notation ϕ as opposed to the notation φ for the nonminimal inflaton). This general framework of the Cauchy problem for the cosmological background and perturbations can be easily extended to include the nonminimal model by reparametrizing the latter to the Einstein frame [11,40], and this will be done in Section 6. Thus, we begin with the action

$$S[g_{\mu\nu},\varphi]=\int d^4x\, g^{1/2}\left(\frac{m_P^2}{16\pi}R(g_{\mu\nu})-\frac{1}{2}(\nabla\phi)^2-V(\phi)\right). \tag{4.1}$$

Under the (unperturbed) ansatz for spatially homogeneous metric (3.22), it takes the minisuperspace form

$$S[a,\phi,N]=\int dt\, Na^3\sqrt{\gamma}\left[\frac{3}{\kappa}\left(\frac{1}{a^2}-\frac{\dot a^2}{N^2a^2}\right)+\frac{1}{2}\frac{\dot\phi^2}{N^2}-V(\phi)\right], \tag{4.2}$$

$$\sqrt{\gamma}\equiv 2\pi^2,\qquad \kappa=\frac{8\pi}{m_P^2}. \tag{4.3}$$

Classical equations for this action in the cosmic time gauge, $N=1$, read

$$\frac{1}{a^3\sqrt{\gamma}}\frac{\delta S}{\delta N}\equiv\frac{3}{\kappa}\left(\frac{1}{a^2}+\frac{\dot a^2}{a^2}\right)-\frac{\dot\phi^2}{2}-V(\phi)=0, \tag{4.4}$$

$$\frac{1}{Na^3\sqrt{\gamma}}\frac{\delta S}{\delta\phi}\equiv-\ddot\phi-3\frac{\dot a}{a}\dot\phi-V_\phi(\phi)=0, \tag{4.5}$$

$$\frac{1}{3Na^2\sqrt{\gamma}}\frac{\delta S}{\delta a}\equiv\frac{1}{\kappa}\left(\frac{1}{a^2}+2\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right)+\frac{\dot\phi^2}{2}-V(\phi)=0. \tag{4.6}$$

The first of Eqs. (5.1) represents the nondynamical Hamiltonian constraint. In terms of the momenta conjugated to a and ϕ, $\Pi_\phi=\sqrt{\gamma}a^3\dot\phi/N$, $\Pi_a=-6\sqrt{\gamma}a\dot a/\kappa N$, this constraint has the following form

$$\boldsymbol{H}(a,\phi,\Pi_a,\Pi_\phi)=-\frac{\kappa}{12a\sqrt{\gamma}}\Pi_a^2+\frac{1}{2a^3\sqrt{\gamma}}\Pi_\phi^2+a^3\sqrt{\gamma}\left[V(\phi)-\frac{3}{\kappa a^2}\right]=0, \tag{4.7}$$

which at the quantum level in the coordinate representation of the quantum minisuperspace, $\widehat{\Pi}_a=\partial/i\partial a$, $\widehat{\Pi}_\phi=\partial/i\partial\phi$, gives rise to the minisuperspace Wheeler–DeWitt equation on

$\boldsymbol{\Psi}(\phi, a)$

$$\boldsymbol{H}(a, \phi, \partial/i\partial a, \partial/i\partial\phi)\boldsymbol{\Psi}(\phi, a) = 0. \tag{4.8}$$

There are two well known semiclassical solutions of this equation — the so-called no-boundary and tunneling wavefunctions. In the approximation of the inflationary slow roll (when the derivatives with respect to ϕ are much smaller than the derivatives with respect to a) these two solutions read [28]

$$\boldsymbol{\Psi}_{\mathrm{NB}}(\phi, a) = C_{\mathrm{NB}}\big(a^2H^2(\phi) - 1\big)^{-1/4} \exp\left[-\frac{1}{2}I(\phi)\right] \cos\left[S(a, \phi) + \frac{\pi}{4}\right], \tag{4.9}$$

$$\boldsymbol{\Psi}_{\mathrm{T}}(\phi, a) = C_{\mathrm{T}}\big(a^2H^2(\phi) - 1\big)^{-1/4} \exp\left[+\frac{1}{2}I(\phi) + iS(a, \phi) + \frac{i\pi}{4}\right]. \tag{4.10}$$

They describe two types of the nucleation of the Lorentzian quasi-de Sitter spacetime (described by the Hamilton–Jacobi function $S(\phi, a)$) from the gravitational semi-instanton — the Euclidean signature hemisphere bearing the Euclidean gravitational action $I(\phi)/2$

$$I(\phi) = -\frac{\pi m_P^2}{H^2(\phi)},$$
$$S(\phi, a) = -\frac{\pi m_P^2}{2H^2(\phi)}\big(a^2H^2(\phi) - 1\big)^{3/2}. \tag{4.11}$$

The size of this hemisphere — its inverse radius — as well as the curvature of the quasi-de Sitter spacetime are determined by the effective Hubble constant, $\dot{a}/a \simeq H(\phi)$, driving the inflationary dynamics of the model

$$H^2(\phi) = \frac{8\pi V(\phi)}{3m_P^2} = \frac{\kappa V(\phi)}{3}. \tag{4.12}$$

The nucleation of the Lorentzian spacetime from the Euclidean hemisphere takes place at $a = 1/H(\phi)$. This domain forms the one-dimensional curve in the two-dimensional superspace. Its embedding equation can be written in the form

$$\chi(\phi, a) = a - \sqrt{\frac{3}{\kappa V(\phi)}} = 0. \tag{4.13}$$

The dimensionality of this subspace coincides with the number of physical degrees of freedom in the minisuperspace sector of the model. The intrinsic coordinate on this subspace becomes the physical coordinate and the restriction of the Dirac wavefunction $\boldsymbol{\Psi}(\phi, a)$ to this subspace becomes the physical wavefunction, provided one takes care of a proper relation between the quantum measures on the original superspace and the physical subspace. For a generic constrained system, the details of such a quantum reduction can be found in [24,32,33]. Here we just briefly repeat it for our model.

Let us identify $\chi(\phi, a)$ in (4.13) with the gauge condition fixing the time reparametrization invariance in the theory (4.2) and choose ϕ as the physical coordinate. Then, according to the formalism of [32], the physical wavefunction $\boldsymbol{\Psi}(\phi)$ in the one-loop (linear in $\hbar$ approximation) can be obtained from the semiclassical Dirac wavefunction $\boldsymbol{\Psi}(\phi, a)$ by the transformation

$$\boldsymbol{\Psi}(\phi) = \big|\{\chi, \boldsymbol{H}\}\big|^{1/2} \boldsymbol{\Psi}(\phi, a)\big|_{\chi(\phi,a)=0}. \tag{4.14}$$

Here we distinguish the original Dirac wavefunction in 2-dimensional minisuperspace from the physical wavefunction by the number of their arguments. The factor $|\{\chi, \boldsymbol{H}\}|$ — the Poisson bracket of the gauge condition with the first class Hamiltonian constraint — is the Faddeev–Popov determinant which should be calculated at the semiclassical values of momenta, $\Pi_a = \partial_a S(\phi, a)$, $\Pi_\phi = \partial_\phi S(\phi, a)$. In the slow roll approximation, when the Π_ϕ-momentum is negligible, this factor equals $|\{\chi, \boldsymbol{H}\}|^{1/2} \sim (H^2 a^2 - 1)^{1/4}$ and, thus, cancels the preexponential factors in Eqs. (4.9), (4.10) divergent at the nucleation surface (4.13).

Thus, the physical wavefunction on the nucleation surface (4.13) which should be regarded as *the Cauchy surface in minisuperspace* reads as

$$\boldsymbol{\Psi}_{\mathrm{NB,T}}(\phi) = C_{\mathrm{NB,T}} \exp\left[\mp \frac{1}{2} I(\phi)\right], \tag{4.15}$$

minus and plus signs related respectively to the the no-boundary and tunneling states. It is well known that the graphs of these wavefunctions are very flat for the situations when the slow roll approximation holds (equivalent to small ϕ-derivatives). Therefore, they are generally not normalizable and do not have good probability peaks that could be interpreted as a source of initial conditions for inflation. The inclusion of loop terms via Eq. (2.4) might lead to the normalizability of the wavefunction and, for the model of the nonminimally coupled inflaton, even yield a sharp probability peak of the above type. Then, the expectation value of the inflaton $\phi = \langle\hat{\phi}\rangle$ becomes finite. It is determined by the location of this peak and serves as the initial condition for the classical extremal that will be used as the background for the calculation of the one-loop radiation currents.

The second initial condition for this classical extremal — the time derivative of the inflaton — arises from the the expectation value of the *physical* momentum conjugated to ϕ, $p_\phi = \langle\hat{p}_\phi\rangle$. In view of reality of the initial density matrix (3.50) this expectation value is vanishing

$$\langle\hat{p}_\phi\rangle = \int d\phi \, \frac{1}{i} \frac{\partial}{\partial\phi} \rho(\phi, \phi')\big|_{\phi'=\phi} = 0. \tag{4.16}$$

From the Hamiltonian reduction of the symplectic form in the gauge $\chi(\phi, a) = 0$ it follows that the physical momentum expresses in terms of the original momenta

$$\begin{aligned} &\Pi_a \, da + \Pi_\phi \, d\phi = p_\phi \, d\phi, \\ &p_\phi = \Pi_\phi - \Pi_a \chi_\phi / \chi_a, \quad \chi_\phi \equiv \partial_\phi \chi, \quad \chi_a \equiv \partial_a \chi. \end{aligned} \tag{4.17}$$

Therefore, for $p_\phi = 0$, Π_ϕ homogeneously expresses in terms of Π_a and, after plugging this relation into the Hamiltonian constraint (4.7), it implies that at the initial Cauchy surface $\Pi_\phi = 0$ and $\Pi_a = 0$. Thus, the full set of initial conditions for the classical background reads

$$\phi = \langle\hat{\phi}\rangle, \qquad a = \frac{1}{H(\phi)}, \qquad \dot{\phi} = \dot{a} = 0. \tag{4.18}$$

5. Cauchy problem for cosmological perturbations

In this section we pose the Cauchy problem for quantum cosmological perturbations propagating on the classical background of the previous section. First, the set of perturbations is reduced by the technique of [20] to the set of linearized invariants of spacetime diffeomorphisms, and their quadratic action is constructed. The ghost nature of their minisuperspace sector is revealed and the original perturbations are built in terms of invariants in the Newton gauge. Then, quantum initial conditions for perturbations are obtained with the aid of the linearized version of the minisuperspace gauge introduced above. Again, we consider the minimal model which will be later, in Section 6, reparametrized to the nonminimal curvature coupling.

5.1. Hamiltonian reduction to the physical sector

Here we start with the physical reduction for cosmological perturbations on the classical background of Section 4. In the main, we follow the notations of [20] where this reduction was presented in much detail. In particular, we use the conformal time denoted by η corresponding to $N = a(\eta)$. In this gauge the classical equations of motion (4.4)–(4.6) have the form

$$\begin{aligned}
&\frac{3}{\kappa}\mathcal{H}^2 - \frac{\phi'^2}{2} + \frac{3}{\kappa} - a^2 V = 0,\\
&\mathcal{H}^2 + 1 - \mathcal{H}' = \frac{\kappa}{2}\phi'^2,\\
&\phi'' + 2\mathcal{H}\phi' + a^2 V_\phi = 0,
\end{aligned} \tag{5.1}$$

where primes denote the derivatives with respect to the conformal time, subscript ϕ implies the partial derivative with respect to the inflaton, $V_\phi \equiv \partial_\phi V(\phi)$, and $\mathcal{H}$ is the "conformal" Hubble constant

$$\mathcal{H} \equiv \frac{a'}{a}, \qquad a' \equiv \frac{da}{d\eta}, \tag{5.2}$$

related to the Hubble constant in cosmic time H by the equation $\mathcal{H} = aH$.

The cosmological perturbations $(h_{ij}, A, S_i, \delta\phi)$ of metric and inflaton field are introduced according to the ansatz

$$ds^2_{\text{total}} = a^2(\eta)\big[-(1+2A)\,d\eta^2 + 2S_i\,dx^i\,d\eta + (\gamma_{ij} + h_{ij})\,dx^i\,dx^j\big], \tag{5.3}$$

$$\phi_{\text{total}} = \phi + \delta\phi, \tag{5.4}$$

$$h_{ij} = -2\psi\gamma_{ij} + 2E_{|ij} + 2F_{(i|j)} + t_{ij}, \tag{5.5}$$

$$S_i = \nabla_i B + V_i, \qquad \nabla_i F^i = \nabla_i V^i = t^i_i = \nabla^i t_{ij} = 0.$$

They consist of the scalar perturbations $(\psi, \delta\phi, E, A, B)$, transverse vector perturbations (F_i, V_i) and transverse-traceless tensor ones t_{ij}. Here ∇_i denotes the spatial covariant derivative.

Spatially homogeneous modes from the minisuperspace sector, upon which we focuse in this paper, belong to scalar perturbations. As discussed above, the inhomogeneous modes which contribute to the J^f radiation current can be treated by the effective action method and do not require a manifest physical reduction. Thus we consider only the scalar sector. After constructing the quadratic part of the action in terms of scalar perturbations one introduces the momenta conjugated to $(\psi, \delta\phi, E)$

$$\begin{aligned} \Pi_\psi &= \frac{2a^2\sqrt{\gamma}}{\kappa}\left[-3\left(\psi' - \frac{\kappa}{2}\phi'\delta\phi + \mathcal{H}A\right) - \Delta(B - E')\right], \\ \Pi_{\delta\phi} &= a^2\sqrt{\gamma}(\delta\phi' - \phi' A), \\ \Pi_E &= \frac{2a\sqrt{\gamma}}{\kappa}\Delta\left[\psi' - \frac{\kappa}{2}\phi'\delta\phi + \mathcal{H}A - (B - E')\right], \end{aligned} \tag{5.6}$$

and finds out that (A, B) play the role of Lagrange multipliers to the linearized Hamiltonian and momentum constraints

$$C_A = -\mathcal{H}\Pi_\psi + \phi'\Pi_{\delta\phi} + a^2\sqrt{\gamma}\left[-\frac{2}{\kappa}D\psi + (\mathcal{H}\phi' - \phi'')\delta\phi\right], \tag{5.7}$$

$$C_B = \Pi_E, \tag{5.8}$$

where D is the following modified covariant Laplacian acting on a closed 3-sphere with the metric γ_{ij}

$$D = \Delta + 3, \quad \Delta = \gamma^{ij}\nabla_i\nabla_j. \tag{5.9}$$

The constraints (5.7) generate the diffeomorphisms in the scalar perturbation sector with respect to the vector-field parameter $\lambda^\mu = (\lambda^0, \nabla^i\lambda)$

$$\delta_\lambda\psi = -\mathcal{H}\lambda^0, \qquad \delta_\lambda(\delta\phi) = \phi'\lambda^0, \qquad \delta_\lambda E = \lambda, \tag{5.10}$$

$$\delta_\lambda\Pi_\psi = \frac{2a^2\sqrt{\gamma}}{\kappa}D\lambda^0, \qquad \delta_\lambda\Pi_{\delta\phi} = a^2\sqrt{\gamma}(\phi'' - \mathcal{H}\phi')\lambda^0, \qquad \delta_\lambda\Pi_E = 0, \tag{5.11}$$

accompanied by the transformations of the Lagrange multipliers $\delta_\lambda A = (\lambda^0)' + \mathcal{H}\lambda^0$, $\delta_\lambda B = \lambda' - \lambda^0$. There are two obvious invariants of the gauge canonical transformations (5.10), (5.11)

$$\boldsymbol{\Psi} = \psi + \frac{\mathcal{H}}{\phi'}\delta\phi, \tag{5.12}$$

$$\boldsymbol{\Pi}_\psi = \Pi_\psi - \frac{2a^2\sqrt{\gamma}}{\kappa\phi'}D\delta\phi. \tag{5.13}$$

It turns out that after solving the constraints, $C_A = C_B = 0$, (5.7), (5.8), with respect to $\Pi_{\delta\phi}$ and Π_E and feeding the result into the canonical action the latter entirely expresses in terms of these two invariants. Moreover, they play the role of a single pair of canonically conjugated variables in the physical sector [20]: on the constraint surface in phase space the original symplectic form goes over into the physical one, $\Pi_\psi\psi' + \Pi_{\delta\phi}\delta\phi' + \Pi_E E' = \boldsymbol{\Pi}_\psi\boldsymbol{\Psi}' + (\ldots)'$. The corresponding canonical action quadratic in $(\boldsymbol{\Psi}, \boldsymbol{\Pi}_\psi)$ reads

$$S[\boldsymbol{\Psi}, \boldsymbol{\Pi}_\psi]\big|_2 = \int d\eta\left[\boldsymbol{\Pi}_\psi\boldsymbol{\Psi}' - \frac{2a^2\sqrt{\gamma}}{\kappa^2\phi'^2}\left(D\boldsymbol{\Psi} + \frac{\kappa\mathcal{H}}{2a^2\sqrt{\gamma}}\boldsymbol{\Pi}_\psi\right)^2\right.$$

$$-\frac{a^2\sqrt{\gamma}}{\kappa}\boldsymbol{\Psi}D\boldsymbol{\Psi}+\frac{\kappa}{4a^2\sqrt{\gamma}}\boldsymbol{\Pi}_\psi\frac{1}{D}\boldsymbol{\Pi}_\psi\bigg]. \tag{5.14}$$

Somewhat simpler form this action acquires in terms of the new variables $(\boldsymbol{q},\boldsymbol{p})$ related to (5.12), (5.13) by the canonical transformation

$$\boldsymbol{q}=\frac{2a}{\kappa\phi'}\boldsymbol{\Psi}+\frac{\mathcal{H}}{\phi'}\frac{1}{a\sqrt{\gamma}D}\boldsymbol{\Pi}_\psi,$$
$$\boldsymbol{p}=-\frac{\phi' a}{2\mathcal{H}}\sqrt{\gamma}D\boldsymbol{\Psi}+\frac{\kappa\phi'}{4a}\boldsymbol{\Pi}_\psi. \tag{5.15}$$

In terms of them the quadratic action in the *physical* sector of scalar perturbations looks as

$$S[\boldsymbol{q},\boldsymbol{p}]\big|_2=\int d\eta\bigg[\boldsymbol{p}\boldsymbol{q}'+\boldsymbol{p}\boldsymbol{q}\bigg(\frac{\phi''}{\phi'}+\frac{\kappa\phi'^2}{4\mathcal{H}}\bigg)+\frac{1}{2\sqrt{\gamma}}\boldsymbol{p}\frac{1}{D}\boldsymbol{p}$$
$$+\frac{\kappa\phi'^2}{8\mathcal{H}^2}\bigg(-\mathcal{H}^2+1-\frac{\kappa\phi'^2}{4}\bigg)\sqrt{\gamma}\,\boldsymbol{q}D\boldsymbol{q}-\frac{1}{2}\sqrt{\gamma}(D\boldsymbol{q})^2\bigg]. \tag{5.16}$$

With the extremal expression for the momentum

$$\boldsymbol{p}=-\sqrt{\gamma}D\big[\boldsymbol{q}'+\big(\phi''/\phi'+\kappa\phi'^2/4\mathcal{H}\big)\boldsymbol{q}\big] \tag{5.17}$$

the Lagrangian form of this action is even shorter

$$S[\boldsymbol{q}]\big|_2=\frac{1}{2}\int d\eta\,\sqrt{\gamma}(-D\boldsymbol{q})\bigg[-\frac{d^2}{d\eta^2}+\phi'\bigg(\frac{1}{\phi'}\bigg)''+\frac{\kappa\phi'^2}{2}+D\bigg]\boldsymbol{q}. \tag{5.18}$$

The invariant field $\boldsymbol{q}$ here is well known from the theory of cosmological perturbations [20]. It is actually given by the so-called Bardeen variable [20,36], $\boldsymbol{\Phi_H}=\mathcal{H}(B-E')-\psi$, $\boldsymbol{q}=-2a\boldsymbol{\Phi_H}/\kappa\phi'$.

Note that the operator D given by (5.9) is negative definite except for two modes: the zero mode corresponding to the Laplacian eigenvalue $\Delta=-3$ and the spatially homogeneous mode for $\Delta=0$, $D=+3$. In view of the overall factor $-D$ the zero mode does not enter the action at all, while the homogeneous mode enters (5.18) with a wrong sign — its kinetic term is negative. Thus, this is a ghost variable signifying the classical instability of the model. This instability at the linearized level is nothing but the manifestation of the inflation which is a huge instability phenomenon incorporating the runaway modes. In contrast with a conventional wisdom of the S-matrix theory, this instability should not be regarded as an irrecoverable flaw of the theory, because we know a nonlinear damping mechanism that provides an exit from the inflation stage in case of the inflaton field rolling down to smaller values of the potential. In particular, no special measures like introducing the indefinite metric should be undertaken to eradicate this phenomenon. Homogeneous fluctuations of the inflaton field do not have a particle nature and one should not take care of guaranteeing the energy positivity of their excitations. Therefore, this mode can and should be quantized in the coordinate representation with positive metric in the Hilbert space.

A single spatially homogeneous mode $\boldsymbol{q}(\eta)$ contained in the full set of

$$\boldsymbol{q}(x)=\big(\boldsymbol{q}(\eta),\boldsymbol{q}(\eta,\mathbf{x})\big) \tag{5.19}$$

corresponds to the $D = +3$ eigenvalue of the operator (5.9) in the action (5.18). It also satisfies all the above relations with a simple ultralocal substitution $D = +3$ and is actually responsible for the perturbations in the minisuperspace sector of the cosmological model. Indeed, from the metric ansatz (5.3), (5.5) it follows that spatially homogeneous variables $\psi(\eta)$, $\delta\phi(\eta)$ and $A(\eta)$ induced by $\boldsymbol{q}(\eta)$ generate the variations of the scale factor, inflaton field and lapse function

$$\delta a = -a\psi + \mathrm{O}\big(\psi^2\big), \qquad \delta\phi, \qquad \delta N = aA + \mathrm{O}\big(A^2\big). \tag{5.20}$$

Actual expression for $\psi(\eta)$, $\delta\phi(\eta)$ and $A(\eta)$ in terms of $(\boldsymbol{q}(\eta), \boldsymbol{p}(\eta))$ depend on the particular gauge chosen for minisuperspace variables. In what follows we will need two types of such gauges. One will be used for gauge fixing the dynamical evolution of perturbations as a function of dynamically evolving invariant variable $\boldsymbol{q}(\eta)$. Another gauge serves as a part of the quantum Cauchy problem — as shown in the previous section, it facilitates the quantum reduction to the physical sector and relates the wavefunction to the initial conditions for both the classical background and the *homogeneous* perturbation variable $\boldsymbol{q}(\eta)$. The first gauge may coincide with the second one. However, its use is strongly biased by practical necessities of the theory of cosmological perturbations [37] and, therefore, is usually chosen to be the Newton gauge which is essentially different from the minisuperspace gauge of Section 3. Thus we consider these two gauges separately.

5.2. Newton gauge

Newton gauge is widely used in the theory of cosmological perturbations [37] to express them in terms of the Bardeen invariant $\boldsymbol{q}$. The Newton gauge for spatially inhomogeneous modes reads

$$B = 0, \qquad E = 0. \tag{5.21}$$

From the equations for momenta (5.6) and the momentum constraint $C_B = 0$ this implies that

$$\varPi_\psi = -\frac{6a^2\sqrt{\gamma}}{\kappa}\left(\psi' - \frac{\kappa}{2}\phi'\delta\phi + \mathcal{H}A\right) = 0. \tag{5.22}$$

In the spatially homogeneous sector of the theory, where the contribution of B and E is missing (they enter only differentiated with respect to spatial coordinates) the latter equation should be regarded as the definition of the Newton gauge. This gauge involves only the phase space coordinate — the momentum $\varPi_\psi$ — and, therefore, it is unitary and can be identified with the gauge (3.13).

Canonical equations of motion for $(\boldsymbol{\Psi}, \boldsymbol{\varPi}_\psi)$, which follow from the action (5.14), in this gauge have a simple corollary $(a\psi)'/a - \kappa\phi'\delta\phi/2 = 0$. When compared with (5.6), $\varPi_\psi = 0$, this corollary yields the main relation in the Newton gauge

$$A = \psi. \tag{5.23}$$

Then one easily expresses all the perturbations in terms of the physical phase space variables $(\boldsymbol{q}, \boldsymbol{p})$. On substituting the Lagrangian value of the momentum (5.17) these expressions finally simplify to

$$\psi = \frac{\kappa\phi'}{2a}\boldsymbol{q}, \tag{5.24}$$

$$\delta\phi = \frac{(\phi'\boldsymbol{q})'}{a\phi'}. \tag{5.25}$$

Eqs. (5.23)–(5.25) form the needed set of relations (3.16) of the physical reduction for minisuperspace perturbations.

5.3. Minisuperspace gauge

The minisuperspace gauge $\chi(a,\phi) = 0$ of Section 3 was used for the physical reduction of the minisuperspace wavefunction and for establishing the tree-level initial conditions — for the classical background. Let us now use it in order to find the initial conditions for $(\boldsymbol{q}, \boldsymbol{p})$. The linearized minisuperspace gauge condition (the gauge (3.13) in condensed notations of Section 3.1) gives the perturbation of a in terms of $\delta\phi$. Taking into account the relations (5.20), expressing δa in terms of the perturbation ψ, we get

$$\psi = \frac{\chi_\phi}{a\chi_a}\delta\phi. \tag{5.26}$$

The corresponding reduction of the symplectic form gives the expression for the physical momentum $p_{\delta\phi}$ conjugated to $\delta\phi$

$$\Pi_\psi\psi' + \Pi_{\delta\phi}\delta\phi' = p_{\delta\phi}\delta\phi' + \cdots, \qquad p_{\delta\phi} = \Pi_{\delta\phi} + \Pi_\psi\frac{\chi_\phi}{a\chi_a}. \tag{5.27}$$

Then, by solving the linearized constraint $C_A = 0$ one easily finds Π_ψ and $\Pi_{\delta\phi}$ as functions of $(\delta\phi, p_{\delta\phi})$ and, via the formalism above, proceeds to the final transformation relating $(\boldsymbol{q}, \boldsymbol{p})$ to $(\delta\phi, p_{\delta\phi})$

$$\boldsymbol{q}(\eta) = \frac{1}{M}\left[\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\frac{a^2\chi_a\mathcal{H}}{3} + \frac{2\chi_\phi^2}{\kappa\chi_a}\right]\delta\phi(\eta) + \frac{\chi_a\mathcal{H}}{3M\sqrt{\gamma}}p_{\delta\phi}(\eta), \tag{5.28}$$

$$\boldsymbol{p}(\eta) = \sqrt{\gamma}\left\{\frac{\kappa\phi'^2}{4M}\left[\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\frac{a^2\chi_a\mathcal{H}}{3} + \frac{2\chi_\phi^2}{\kappa\chi_a}\right] - \frac{M}{\chi_a}\right\}\frac{3}{\mathcal{H}}\delta\phi(\eta) + \frac{\chi_a}{M}\frac{\kappa\phi'^2}{4}p_{\delta\phi}(\eta), \tag{5.29}$$

$$M \equiv \phi'\chi_\phi + \mathcal{H}a\chi_a. \tag{5.30}$$

Note that this relation is written down in the homogeneous sector which is emphasized by the time arguments of the phase space variables. In the right-hand side of these equations the spatial homogeneity manifests itself in the particular value of the operator D, $D = 3$. One can easily check that this transformation is canonical, $\{\boldsymbol{q}, \boldsymbol{p}\} = \{\delta\phi, p_{\delta\phi}\} = 1$, and invertible. Inverting it, one can find all the minisuperspace perturbations as functions of $\boldsymbol{q}$ and $\boldsymbol{q}'$, similarly to the relations (5.23)–(5.25) in the Newton gauge. However, the goal of working in the gauge (4.13) is somewhat different. We shall need the relations (5.28)–(5.30) in order to express initial conditions for $(\boldsymbol{q}(\eta), \boldsymbol{p}(\eta))$ in terms of initial conditions for $(\delta\phi, p_{\delta\phi})$. The latter in their turn follow from the cosmological wavefunction in the physical sector.

We begin by noting that at the initial moment of time the following relations hold

$$\mathcal{H} = \eta + \mathrm{O}(\eta^2), \qquad \phi' = -\frac{3V_\phi}{\kappa V}\eta + \mathrm{O}(\eta^2), \qquad \frac{\mathcal{H}}{\phi'}\bigg|_0 = \frac{\mathcal{H}'}{\phi''}\bigg|_0 = -\frac{\kappa V}{3V_\phi},$$
$$M = \eta\left(\frac{3}{\kappa V}\right)^{1/2}\left(1 - \frac{3}{2}\frac{V_\phi^2}{\kappa V^2}\right) + \mathrm{O}(\eta^2), \quad \eta \to 0.$$

Using these relations in Eqs. (5.28), (5.29) we obtain the asymptotic behaviour of the invariant variables $(\boldsymbol{q}, \boldsymbol{p})$ for $\eta \to 0$ in terms of the physical variables of the minisuperspace gauge fixing $(\delta\phi(\eta), p_{\delta\phi}(\eta))$

$$\left(\frac{\kappa V}{3}\right)^{1/2}\boldsymbol{q} \simeq -\frac{1}{3\eta}\delta\phi + \frac{\kappa V}{9}\frac{1}{1-9\epsilon^2/16}\frac{p_{\delta\phi}}{\sqrt{\gamma}}, \tag{5.31}$$

$$\left(\frac{3}{\kappa V}\right)^{1/2}\boldsymbol{p} \simeq -\frac{9}{\kappa V}\left(1 - 9\epsilon^2/32\right)\sqrt{\gamma}\,\delta\phi, \quad \eta \to 0. \tag{5.32}$$

Just to emphasize the role of the slow roll expansion we retained here the corrections proportional to the smallness parameter

$$\epsilon^2 \equiv \frac{8}{3}\frac{V_\phi^2}{\kappa V^2} \ll 1. \tag{5.33}$$

In what follows we shall systematically discard such corrections retaining only the leading order of the slow roll expansion.

Important peculiarity of the behaviour (5.31), (5.32) is its singularity at $\eta \to 0$. This singularity is, however, an artifact of the definition of the invariant variables (5.15) nonanalytic at $\phi' \to 0$, rather than the manifestation of some physical inconsistencies. To see it, one can decompose the general classical solution for $\boldsymbol{q}(\eta)$ in the sum of two linearly independent solutions of the equation of motion for the action (5.18)

$$\boldsymbol{q}(\eta) = c_+\boldsymbol{q}_+(\eta) + c_-\boldsymbol{q}_-(\eta), \tag{5.34}$$

$$\left(-\frac{d^2}{d\eta^2} + \phi'\left(\frac{1}{\phi'}\right)'' + \frac{\kappa\phi'^2}{2} + 3\right)\boldsymbol{q}_\pm(\eta) = 0. \tag{5.35}$$

Because of $\phi'(1/\phi')'' \sim 2/\eta^2$, $\eta \to 0$, the initial moment $\eta = 0$ is a singular point of this differential equation, at which one of the two solutions, $\boldsymbol{q}_-(\eta)$, diverges as $1/\phi'$. One can make a singular rescaling,

$$\boldsymbol{q} \equiv \frac{\boldsymbol{Q}}{\phi'}, \tag{5.36}$$

to a new variable $\boldsymbol{Q}(\eta)$ which is finite at this point. It satisfies the equation

$$\left(-\frac{d^2}{d\eta^2} + 2\frac{\phi''}{\phi'}\frac{d}{d\eta} + \frac{\kappa\phi'^2}{2} + 3\right)\boldsymbol{Q}(\eta) = 0, \tag{5.37}$$

and has as two solutions the following regular functions

$$\boldsymbol{Q}_+(\eta) = \eta^3\big(1 + \mathrm{O}\big(\eta^2\big)\big), \tag{5.38}$$

$$\boldsymbol{Q}_-(\eta) = 1 - 3\eta^2/2 + \mathrm{O}\big(\eta^3\big). \tag{5.39}$$

Substituting the decomposition (5.34), with $\boldsymbol{q}_\pm$ related to $\boldsymbol{Q}_\pm$ by (5.36), to the left-hand sides of (5.31), (5.32) one obtains the system of equations for $c_\pm$ with singular coefficients. This system, however, has a regular solution in terms of the initial conditions for physical variables $(\delta\phi(0),\, p_{\delta\phi}(0))$

$$c_+ = -\frac{1}{3}\left(\frac{\kappa V}{3}\right)^{1/2}\frac{V_\phi}{\kappa V}\frac{p_{\delta\phi}(0)}{\sqrt{\gamma}}, \tag{5.40}$$

$$c_- = \left(\frac{3}{\kappa V}\right)^{1/2}\frac{V_\phi}{\kappa V}\delta\phi(0). \tag{5.41}$$

This basic relation will be used throughout the rest of the paper to express the Heisenberg operators of quantum perturbations $\Delta\widehat{Q}_{\mathrm{phys}}(\eta)$ and $\Delta\widehat{Q}(\eta)$ in terms of the Schrödinger operators, $\delta\hat{\phi}(0) = \delta\phi$, $\hat{p}_{\delta\phi}(0) = \partial/i\partial(\delta\phi)$, and then find the quantum averages of their bilinear combinations in the inflaton representation of the initial density matrix (3.50).

6. Non-minimal model

In what follows we go over to the model (2.5) that has a good peak-like behaviour of the initial distribution function of the inflaton [6,7,18]. The inflaton–graviton sector of the action in this model can be rewritten in the form

$$S[g_{\mu\nu},\varphi] = \int d^4x\, g^{1/2}\left\{-V(\varphi) + U(\varphi)R - \frac{1}{2}G(\varphi)(\nabla\varphi)^2\right\}. \tag{6.1}$$

In fact, the curvature (and derivative) expansion of any low-energy effective graviton-scalar action can be truncated to this form with some coefficient functions of the zeroth and first order in the curvature — the scalar field potential $V(\varphi)$, the effective φ-dependent Planck "mass" $16\pi U(\varphi)$ and the coefficient of the inflaton kinetic term $G(\varphi)$. In the classical model (2.5) these functions have a particular form

$$U(\varphi) = \frac{m_P^2}{16\pi} + \frac{1}{2}|\xi|\varphi^2, \tag{6.2}$$

$$V(\varphi) = \frac{m^2\varphi^2}{2} + \frac{\lambda\varphi^4}{4}, \tag{6.3}$$

$$G(\varphi) = 1. \tag{6.4}$$

It is well known that the action (6.1) can be transformed to the Einstein frame by a special conformal transformation and reparametrization of the inflaton field $(g_{\mu\nu},\varphi) \to (\bar{g}_{\mu\nu},\bar{\varphi})$,

$$S[g_{\mu\nu},\varphi] = \overline{S}[\bar{g}_{\mu\nu},\bar{\varphi}],$$

$$\overline{S}[\bar{g}_{\mu\nu},\bar{\varphi}] = \int d^4x\, \bar{g}^{1/2}\left\{-\overline{V}(\bar{\varphi}) + \frac{m_P^2}{16\pi}R(\bar{g}_{\mu\nu}) - \frac{1}{2}(\overline{\nabla}\bar{\varphi})^2\right\}. \tag{6.5}$$

In what follows, we shall denote the fields and other objects in the Einstein frame of the nonminimal model by bars and identify them with those of the minimal model considered in Sections 3–5. In this way we reduce all the calculations, Cauchy data setting, gauge fixing, reduction to the physical sector, etc. to the algorithms derived above for the case of the minimal model.

6.1. Reparametrization to the minimal frame

The transformations relating the actions $S[g_{\mu\nu},\varphi]$ and $\overline{S}[\bar{g}_{\mu\nu},\bar{\varphi}]$ are implicitly given by equations [11,40]

$$\bar{g}_{\mu\nu} = \frac{16\pi U(\varphi)}{m_P^2} g_{\mu\nu}, \tag{6.6}$$

$$\left(\frac{d\bar{\varphi}}{d\varphi}\right)^2 = \frac{m_P^2}{16\pi}\frac{GU + 3U_\varphi^2}{U^2}, \tag{6.7}$$

where, similarly to previous sections, φ-subscripts denote the derivatives of the coefficient functions with respect to the inflaton, $V_\varphi \equiv dV/d\varphi$, $V_{\varphi\varphi} \equiv d^2V/d\varphi^2$, etc. The action in terms of new fields (6.5) has a minimal coupling and the new inflaton potential

$$\overline{V}(\bar{\varphi}) = \left(\frac{m_P^2}{16\pi}\right)^2 \frac{V(\varphi)}{U^2(\varphi)}\bigg|_{\varphi=\varphi(\bar{\varphi})}. \tag{6.8}$$

For the coefficient functions (6.2), (6.3) the explicit reparametrization between the frames can be found for large value of the nonminimal coupling constant $|\xi| \gg 1$ and small value of the parameter $m_P^2/|\xi|\varphi^2 \ll 1$ [11]

$$\varphi(\bar{\varphi}) \simeq \frac{m_P}{|\xi|^{1/2}}\exp\left[\sqrt{4\pi/3}\left(1+\frac{1}{6|\xi|}\right)^{-1/2}\frac{\bar{\varphi}}{m_P}\right], \tag{6.9}$$

$$\overline{V}(\bar{\varphi}) = \frac{\lambda m_P^4}{256\pi^2|\xi|^2}\left[1 - \frac{1+\delta}{4\pi}\frac{m_P^2}{|\xi|\varphi^2} + \cdots\right]_{\varphi=\varphi(\bar{\varphi})}, \tag{6.10}$$

where we have retained only the first order term in $m_P^2/|\xi|\varphi^2$. In view of (6.9), for large $\bar{\varphi}$ this potential exponentially approaches a constant and satisfies the slow roll approximation with the expansion parameter [11]

$$\epsilon = \frac{m_P}{\sqrt{3\pi}}\frac{\overline{V}_{\bar{\varphi}}(\bar{\varphi})}{\overline{V}(\bar{\varphi})} \simeq \frac{1+\delta}{3\pi}\left(1+\frac{1}{6|\xi|}\right)^{-1/2}\frac{m_P^2}{|\xi|\varphi^2} \ll 1, \tag{6.11}$$

which justifies the smallness of the parameter $m_P^2/|\xi|\varphi^2$ chosen above.

Let us now consider the minimal model of Sections 3–5 as an Einstein frame of the nonminimal model and label all the objects of the minimal model — the metric, inflaton field, scale factor, conformal time, cosmological perturbations and the minisuperspace gauge fixing conditions — by bars

$$\bar{g}_{\mu\nu}, \phi = \bar{\varphi}, \bar{a}, \bar{\eta},$$
$$\bar{\psi}, \delta\phi = \delta\bar{\varphi}, \bar{A}, \bar{\Pi}_\psi, \bar{\Pi}_{\delta\phi}, \bar{\chi}(\bar{a}, \bar{\varphi}), \tag{6.12}$$

as opposed to the objects in the original — nonminimal — frame: $g_{\mu\nu}, \varphi, a, \eta, \psi, \delta\varphi, A$, $\chi(a, \varphi)$. Comparing the metrics in these frames, perturbed by the cosmological disturbances from the scalar sector,

$$ds^2 = a^2\big[-(1+2A)\, d\eta^2 + (1-2\psi)\gamma_{ij}\, dx^i\, dx^j\big],$$
$$d\bar{s}^2 = \bar{a}^2\big[-(1+2\bar{A})d\bar{\eta}^2 + (1-2\bar{\psi})\gamma_{ij}\, dx^i\, dx^j\big], \tag{6.13}$$
$$d\bar{s}^2 = \frac{16\pi U}{m_P^2}\, ds^2, \tag{6.14}$$

one finds the relations between these two sets of variables

$$\bar{a} = \sqrt{\frac{16\pi U(\varphi)}{m_P^2}}a \simeq \sqrt{\frac{8\pi|\xi|\varphi^2}{m_P^2}}a, \qquad \bar{\eta} = \eta, \tag{6.15}$$

$$\delta\phi \equiv \delta\bar{\varphi} = \sqrt{\frac{m_P^2}{16\pi}\frac{U + 3U'^2}{U^2}} \simeq \sqrt{\frac{3}{4\pi}}\frac{m_P}{\varphi}\delta\varphi, \tag{6.16}$$

$$\bar{\psi} = \psi - \frac{U_\varphi}{2U}\delta\varphi \simeq \psi - \frac{\delta\varphi}{\varphi}, \tag{6.17}$$

$$\bar{A} = A + \frac{U_\varphi}{2U}\delta\varphi \simeq A + \frac{\delta\varphi}{\varphi}, \tag{6.18}$$

where the last three relations hold in the linear order of perturbation theory in cosmological disturbances. The canonical momenta $\Pi_{\delta\varphi}$ and Π_ψ obviously transform by the rule contragradient to (6.16) and (6.17).

We also need the frame transformation between the physical sectors defined in the minisuperspace gauge of Section 5.3. The gauge condition itself transforms as a scalar — only in this case it represents one and the same Cauchy surface, written in two different coordinate systems on minisuperspace

$$\bar{\chi}(\bar{a}, \bar{\varphi}) = \chi(a, \varphi). \tag{6.19}$$

As regards the reparametrization between these coordinate systems, (6.9) and (6.15), it has a general form

$$\bar{\varphi} = \bar{\varphi}(\varphi), \qquad \bar{a} = \bar{a}(\varphi, a), \tag{6.20}$$

mixing the inflaton and the scale factor only in the transformation of a. Therefore, the linearized perturbations of a and φ and their momenta in both frames are related by a triangular transformation

$$\delta\bar{\varphi} = \frac{\partial\bar{\varphi}}{\partial\varphi}\delta\varphi, \qquad \bar{\psi} = -\frac{1}{a}\frac{\partial\bar{a}}{\partial\varphi}\delta\varphi + \frac{a}{\bar{a}}\frac{\partial\bar{a}}{\partial a}\psi, \tag{6.21}$$

$$\bar{\Pi}_{\delta\varphi} = \frac{\partial\varphi}{\partial\bar{\varphi}}\Pi_{\delta\varphi} - \frac{1}{a}\frac{\partial a}{\partial\bar{\varphi}}\Pi_\psi, \qquad \bar{\Pi}_\psi = \frac{\bar{a}}{a}\frac{\partial a}{\partial\bar{a}}\Pi_\psi. \tag{6.22}$$

The physical momentum $\bar{p}_{\delta\varphi}$ expresses in terms of phase space momenta by the barred version of Eq. (5.27). Then, in view of the above relations, one easily finds

$$\bar{p}_{\delta\varphi} = \frac{\partial\varphi}{\partial\bar{\varphi}}\, p_{\delta\varphi}. \tag{6.23}$$

This equation holds exactly for an arbitrary choice of the gauge condition function $\chi(a,\varphi)$, and this is a corollary of the triangular form of the transformation (6.20). In our nonminimal model with $|\xi| \gg 1$ this implies the following simple relation between the physical sectors in two frames

$$\delta\bar{\varphi} \simeq \sqrt{\frac{3}{4\pi}} m_P \frac{\delta\varphi}{\varphi}, \qquad \bar{p}_{\delta\varphi} \simeq \sqrt{\frac{4\pi}{3}} \frac{\varphi}{m_P} p_{\delta\varphi}. \tag{6.24}$$

6.2. Quadratic order currents

In the minisuperspace sector of the nonminimal model $Q = (N(t), a(t), \varphi(t))$ the functional derivatives of the classical action read

$$\frac{1}{a^3\sqrt{\gamma}}\frac{\delta S}{\delta N} = 6U(\varphi)\left(\frac{1}{a^2} + \frac{\dot{a}^2}{a^2}\right) + 6U_\varphi(\varphi)\dot{\varphi}\frac{\dot{a}}{a} - \frac{\dot{\varphi}^2}{2} - V(\varphi), \tag{6.25}$$

$$\frac{1}{Na^3\sqrt{\gamma}}\frac{\delta S}{\delta\varphi} = -\ddot{\varphi} - 3\frac{\dot{a}}{a}\dot{\varphi} + 6U_\varphi(\varphi)\left(\frac{1}{a^2} + \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - V_\varphi(\varphi), \tag{6.26}$$

where dots are used to denote the parametrization invariant derivative $\dot{a} \equiv da/Ndt$, $\dot{\varphi} \equiv d\varphi/Ndt$. Now we use the perturbed ansatz (6.14) for total minisuperspace variables in these equations, $N^2 \to N^2_{\rm tot} = a^2(1+2A)$, $a^2 \to a^2_{\rm tot} = a^2(1-2\psi)$, $\varphi \to \varphi_{\rm tot} = \varphi + \delta\varphi$, and carefully expand the first order variations of the classical action up to the second order in perturbations $(A, \psi, \delta\varphi)$ on the *classical* background. The result reads as follows

$$\begin{aligned}
\frac{1}{a\sqrt{\gamma}}\left[\frac{\delta S}{\delta N}\right]_2 ={}& A^2\big(24U\mathcal{H}^2 + 24U_\varphi\mathcal{H}\varphi' - 2\varphi'^2\big)\\
&+ A\psi\big(36U\mathcal{H}^2 + 36U_\varphi\mathcal{H}\varphi' - 3\varphi'^2\big) + A\psi'(24U\mathcal{H} + 12U_\varphi\varphi')\\
&+ A\delta\varphi\big(-12U_\varphi\mathcal{H}^2 - 12U_{\varphi\varphi}\mathcal{H}\varphi'\big) + A\delta\varphi'(-12U_\varphi\mathcal{H} + 2\varphi')\\
&- 12U\psi^2 + \psi\psi'(12U\mathcal{H} + 6U_\varphi\varphi') + 6U\psi'^2\\
&+ \psi\delta\varphi\big[-6U_\varphi\big(1 + 3\mathcal{H}^2\big) - 18U_{\varphi\varphi}\varphi' + 3a^2V_\varphi\big]\\
&+ \psi'\delta\varphi(-12U_\varphi\mathcal{H} - 6U_{\varphi\varphi}\varphi') + \psi\delta\varphi'(-18U_\varphi\mathcal{H} + 3\varphi')\\
&- 6U_\varphi\psi'\delta\varphi' + \delta\varphi^2\left[3U_{\varphi\varphi}\big(1 + \mathcal{H}^2\big) - \frac{1}{2}a^2V_{\varphi\varphi} + 3U_{\varphi\varphi\varphi}\mathcal{H}\varphi'\right]\\
&+ 6U_{\varphi\varphi}\mathcal{H}\delta\varphi\delta\varphi' - \frac{1}{2}\delta\varphi'^2,
\end{aligned} \tag{6.27}$$

$$\begin{aligned}
\frac{1}{a^2\sqrt{\gamma}}\left[\frac{\delta S}{\delta\varphi}\right]_2 ={}& A^2\big(2a^2V_\varphi - 12U_\varphi\big) + AA'(18U_\varphi\mathcal{H} - 3\varphi')\\
&+ A\psi(6a^2V_\varphi - 24U_\varphi) + (A\psi)'(18U_\varphi\mathcal{H} - 3\varphi') + 6U_\varphi(A\psi')'\\
&+ A\delta\varphi\big[6U_{\varphi\varphi}\big(1 - \mathcal{H}^2 - \mathcal{H}'\big) - a^2V_{\varphi\varphi}\big] + 2\mathcal{H}A\delta\varphi'
\end{aligned}$$

$$
\begin{aligned}
&- 6U_{\varphi\varphi}\mathcal{H}A'\delta\varphi + (A\delta\varphi')' \\
&- 12U_{\varphi}\psi^2 + \psi\psi'(18U_{\varphi}\mathcal{H} - 3\varphi') + 6U_{\varphi}\psi\psi'' \\
&+ \psi\delta\varphi\big[-6U_{\varphi\varphi}\big(1 + 3\mathcal{H}^2 + 3\mathcal{H}'\big) + 3a^2 V_{\varphi\varphi}\big] - 18U_{\varphi\varphi}\mathcal{H}\psi'\delta\varphi \\
&+ 6\mathcal{H}\psi\delta\varphi' + 3(\psi\delta\varphi')' - 6U_{\varphi\varphi}\psi''\delta\varphi \\
&+ \delta\varphi^2\left[3U_{\varphi\varphi\varphi}\big(1 + \mathcal{H}^2 + \mathcal{H}'\big) - \frac{1}{2}a^2 V_{\varphi\varphi\varphi}\right].
\end{aligned}
\tag{6.28}
$$

Here the unlabelled variables correspond to the minisuperspace background in the conformal time gauge, $N = a$, $t = \eta$, primes denote derivatives with respect to η and $\mathcal{H} = a'/a$ denotes the conformal time background Hubble constant. The background satisfies classical equations of motion which were used to simplify the coefficients of the above quadratic forms in $\Delta Q = (A, \psi, \delta\varphi)$.

6.3. Quantum rolling force: effective action and minisuperspace contributions

In the presence of the spatial densities of one-loop radiation currents

$$j_N \equiv \frac{1}{a^3\sqrt{\gamma}}\left\langle\left[\frac{\delta S}{\delta N}\right]_2\right\rangle, \tag{6.29}$$

$$j_\varphi \equiv \frac{1}{Na^3\sqrt{\gamma}}\left\langle\left[\frac{\delta S}{\delta \varphi}\right]_2\right\rangle, \tag{6.30}$$

N and φ components of the effective equations of motion in the nonminimal model read

$$6U\left(\frac{1}{a^2} + \frac{\dot a^2}{a^2}\right) + 6U_\varphi\dot\varphi\frac{\dot a}{a} - \frac{\dot\varphi^2}{2} - V + j_N = 0, \tag{6.31}$$

$$-\ddot\varphi - 3\frac{\dot a}{a}\dot\varphi + 6U_\varphi\left(\frac{1}{a^2} + \frac{\ddot a}{a} + \frac{\dot a^2}{a^2}\right) - V_\varphi + j_\varphi = 0. \tag{6.32}$$

In view of Eqs. (3.10) and (3.20) their a-component expresses in terms of the above two ones, so that Eqs. (6.31), (6.32) in a consistent manner exhaust the quantum dynamics of the mean fields. Differentiating the first of them with respect to time one obtains the system of two equations for $\ddot a$ and $\ddot\varphi$. Substituting the solution of this system for $\ddot a$ into the second equation one finally has the equation for the mean inflaton field with quantum contributions to the friction term and the rolling force

$$\ddot\varphi + \left(3\frac{\dot a}{a} - \frac{a}{2\dot a}U_\varphi j_\varphi\right)\dot\varphi - F(\varphi, a, \dot\varphi) = 0, \tag{6.33}$$

$$F(\varphi, a, \dot\varphi) = \frac{2VU_\varphi - UV_\varphi}{U + 3U_\varphi^2} - \frac{1}{2}\dot\varphi^2\frac{d}{d\varphi}\ln\big(U + 3U_\varphi^2\big) + F_{\rm loop}(\varphi, a, \dot\varphi, \dot a), \tag{6.34}$$

$$F_{\rm loop}(\varphi, a, \dot\varphi, \dot a) = \frac{1}{U + 3U_\varphi^2}\left(Uj_\varphi - 2U_\varphi j_N - \frac{a}{2\dot a}\frac{dj_N}{dt}\right). \tag{6.35}$$

The first two terms in Eq. (6.34) represent the classical rolling force, the $\dot\varphi^2$ contribution belonging to the subleading order of the slow roll expansion. As regards the quantum part, its radiation currents in the one-loop approximation split into the contributions of the quantum mechanical sector and the field sector of spatially inhomogeneous modes, $j_{1\text{-loop}} = j^q + j^f$ (cf. Eq. (3.34)). According to the discussion of Section 3.1, see Eqs. (3.45)–(3.48), the f-part of the current can be absorbed by the replacement of the original classical action with the effective one (3.48). This implies the replacement of the classical coefficient functions $V(\varphi), U(\varphi), G(\varphi)$, (6.2)–(6.4), by their effective counterparts

$$S_{\rm eff}[g_{\mu\nu},\varphi] = \int d^4x\, g^{1/2}\left\{-V^{\rm eff}(\varphi) + U^{\rm eff}(\varphi)R - \frac{1}{2}G^{\rm eff}(\varphi)(\nabla\varphi)^2 + \cdots\right\}, \tag{6.36}$$

and truncation of the (generally infinite) series to the first three terms. This truncation is based on two assumptions — the smallness of inflaton derivatives due to the slow roll regime and smallness of $R/m^2_{\rm part}$ — the curvature to particle mass squared ratio. [12] Thus, with this approximation, the effective equations of motion in our nonminimal model take the form of (6.31) and (6.33) with $V_{\rm eff}(\varphi), U_{\rm eff}(\varphi), G_{\rm eff}(\varphi)$ replacing $V(\varphi), U(\varphi), G(\varphi)$ and the radiation currents j_N, j_φ saturated by the contribution of the quantum mechanical mode, j^q_N, j^q_φ. The resulting rolling force in the leading order of the slow roll expansion becomes the sum of the force induced by the effective action, $F^{\rm eff}$, and the quantum mechanical force, F^q,

$$F = F^{\rm eff} + F^q, \tag{6.37}$$

$$F^{\rm eff} = \frac{2V^{\rm eff}U^{\rm eff}_\varphi - U^{\rm eff}V^{\rm eff}_\varphi}{G^{\rm eff}U^{\rm eff} + 3(U^{\rm eff}_\varphi)^2}, \tag{6.38}$$

$$F^q = \frac{1}{U + 3U^2_\varphi}\left(Uj^q_\varphi - 2U_\varphi\, j^q_N - \frac{a}{2\dot a}\frac{d\,j^q_N}{dt}\right). \tag{6.39}$$

The one-loop calculation of $V_{\rm eff}(\varphi), U_{\rm eff}(\varphi)$ and $G^{\rm eff}(\varphi)$ and the effect of $F^{\rm eff}$ on the inflationary dynamics have been studied in [1]. This effect is qualitatively different for the no-boundary and tunneling cases and briefly looks as follows. For the no-boundary state the one-loop corrections in the distribution function add up to form the *full* Euclidean effective action

$$\rho_{\rm NB}(\varphi) = {\rm const}\exp\bigl[-\boldsymbol{\Gamma}(\varphi)\bigr], \tag{6.40}$$

$$\boldsymbol{\Gamma}(\varphi) = \boldsymbol{I}(\varphi) + \boldsymbol{\Gamma}^{1\text{-loop}}(\varphi) = -\frac{96\pi^2[U_{\rm eff}(\varphi)]^2}{V_{\rm eff}(\varphi)} + {\rm O}\bigl(\hbar^2\bigr). \tag{6.41}$$

Its value on the de Sitter instanton follows from that of the classical Euclidean action, $\boldsymbol{I}(\varphi) = -96\pi^2U^2/V$, by replacing the classical coefficient functions $V(\varphi), U(\varphi)$ and $G(\varphi)$ by the effective ones, coinciding with those of the Lorentzian effective action (6.36). Therefore, by the direct inspection of (6.38) one observes that the effective rolling force in

[12] The Schwinger–DeWitt expansion involves the inverse powers of masses of particles of constituent quantum fields. The latter acquire their masses via the Higgs effect due to the interaction with the inflaton, so that this ratio becomes order of magnitude $\lambda/|\xi| \ll 1$ [1].

the *no-boundary* case is proportional to the derivative of the distribution function [13]

$$F^{\rm eff}_{\rm NB} = \frac{1}{96\pi^2 U^{\rm eff}} \frac{(V^{\rm eff})^2}{G^{\rm eff}U^{\rm eff} + 3(U^{\rm eff}_\varphi)^2} \frac{d}{d\varphi} \ln \rho_{\rm NB}(\varphi)$$
$$= -\frac{\lambda m_P^2(1+\delta)}{48\pi\xi^2}\varphi\left(1 - \frac{\varphi^2}{\varphi_I^2}\right) + {\rm O}\big(1/|\xi|^3\big), \tag{6.42}$$

and, thus, vanishes at the probability peak φ_I. The no-boundary peak is realized for $1+\delta < 0$, therefore the point φ_I turns out to be an attractor — quantum terms in effective rolling force lock the inflaton at its constant initial value and give rise to infinitely long inflationary scenario with exactly de Sitter spacetime.

In the tunneling case, the distribution function is not related to the overall effective action, because its tree-level part has a wrong sign. The probability peak exists in the opposite range of the parameter (2.9), $\delta > -1$, and the rolling force

$$F^{\rm eff}_{\rm T} = -\frac{\lambda m_P^2(1+\delta)}{48\pi\xi^2}\varphi\left(1 + \frac{\varphi^2}{\varphi_I^2}\right) + {\rm O}\big(1/|\xi|^3\big) \tag{6.43}$$

has the quantum term which initially doubles the negative classical part. Therefore, the inflaton starts slowly decreasing under the influence of this force, and the tunneling state generates a finite inflation stage with the estimated e-folding number (2.19). These conclusions disregard the contribution of the quantum mechanical radiation currents, and we proceed to their calculation.

6.4. Quantum state of the minisuperspace perturbations and their correlators

The calculation of quantum averages in the quadratic currents (6.27) and (6.28) requires the set of quantum correlators (3.49) of bilinear combinations of minisuperspace disturbances $\Delta\widehat{Q} = (A, \psi, \delta\varphi)$ and their derivatives. For this purpose we, first, need the reduced density matrix of the inflaton field $\rho(\varphi, \varphi')$ in the nonminimal model and, second, the expressions for the Heisenberg operators $\Delta\widehat{Q}(\eta)$ in terms of the Schrödinger operators of initial perturbations and their momenta, $\delta\hat\varphi = \delta\varphi$, $\hat p_{\delta\varphi} = \partial/i\,\partial(\delta\varphi)$.

As we know, the diagonal element of the density matrix has a quasi-gaussian behaviour (2.12), which is, however, insufficient for averaging the operators involving momenta. The necessary off-diagonal elements with one-loop contributions of various massive and massless fields have been calculated in [41,42]. It was shown that in the model with a big $|\xi|$ the initial density matrix describes practically pure quantum state and expresses in terms of the distribution function

$$\rho(\varphi, \varphi')\big|_{t=0} \simeq \sqrt{\rho(\varphi)}\sqrt{\rho(\varphi')}, \qquad |\xi| \gg 1. \tag{6.44}$$

The explanation of this property [41] is based on the fact that the decoherence factor $D(\varphi, \varphi')$ by which the initial density matrix differs from (6.44) is a function of the

[13] The explanation of this observation is simple. In the minimal frame the rolling force is given by the gradient of the potential, while the logarithm of the distribution function is inverse proportional to it, the combination $V^{\rm eff}(\varphi)/[U^{\rm eff}(\varphi)]^2$ representing the minimal frame potential in terms of the nonminimal objects (6.8).

arguments $m/H(\varphi)$ and $m/H(\varphi')$ for a quantum field of a mass m. For large $|\xi|$ the masses of particles generated by the Higgs effect give rise to big and predominantly φ-independent ratio $m/H(\varphi) \sim \sqrt{|\xi|}$, so that $D(\varphi, \varphi') \sim 1$. For massless fields a similar conclusion can be drawn because for them the role of mass is played by the Hubble constant $H(\varphi)$ of the quasi-de Sitter background.

In view of (2.12), the effectively pure quantum state,

$$\boldsymbol{\Psi}_{\mathrm{NB,T}}(\varphi) \simeq \sqrt{\rho_{\mathrm{NB,T}}(\varphi)}, \tag{6.45}$$

in the vicinity of the probability maximum, which is located at φ_I, can, thus, be approximated by the gaussian packet of small quantum width Δ

$$\boldsymbol{\Psi}_{\mathrm{NB,T}}(\delta\varphi) \equiv \boldsymbol{\Psi}_{\mathrm{NB,T}}(\varphi_I + \delta\varphi) = \frac{1}{(2\pi)^{1/4}\sqrt{\Delta}} \exp\left[-\frac{\delta\varphi^2}{4\Delta^2}\right]. \tag{6.46}$$

The operators of quantum disturbances in the $\delta\varphi$-representation, acting on the wavefunction of the above type can be found by collecting together several sets of equations derived above. First, we use Eqs. (6.16)–(6.18), relating ΔQ to the Einstein frame perturbations $\Delta\overline{Q}$. Then, we apply the barred version of Eqs. (5.23)–(5.25) to express the minisuperspace perturbations $\Delta\overline{Q}$ in the Newton gauge as functions of the invariants $\boldsymbol{q}$ and $\boldsymbol{q}'$ in the minimal frame. Finally, we use the set of Eqs. (5.34), (5.36) and (5.40), (5.41) with barred (minimal frame) potential and physical variables to express these invariants in terms of $\delta\bar{\varphi}$ and $\bar{p}_{\delta\varphi}$. The final result looks as follows

$$A = -\frac{1}{3a\varphi'}\sqrt{\frac{\kappa}{|\xi|}}\left[\left(\boldsymbol{Q}'_+ - \frac{3\varphi'}{\varphi}\boldsymbol{Q}_+\right)\hat{c}_+ + \left(\boldsymbol{Q}'_- - \frac{3\varphi'}{\varphi}\boldsymbol{Q}_-\right)\hat{c}_-\right], \tag{6.47}$$

$$\psi = \frac{1}{3a\varphi'}\sqrt{\frac{\kappa}{|\xi|}}\left[\left(\boldsymbol{Q}'_+ + \frac{3\varphi'}{\varphi}\boldsymbol{Q}_+\right)\hat{c}_+ + \left(\boldsymbol{Q}'_- + \frac{3\varphi'}{\varphi}\boldsymbol{Q}_-\right)\hat{c}_-\right], \tag{6.48}$$

$$\delta\varphi = \frac{1}{\sqrt{6|\xi|}a\varphi'}\left(\boldsymbol{Q}'_+\hat{c}_+ + \boldsymbol{Q}'_-\hat{c}_-\right), \tag{6.49}$$

where the operators $\hat{c}_\pm$ with the aid of (6.24) read as

$$\hat{c}_+ = -\frac{1+\delta}{576\pi^3}\sqrt{\frac{2\pi}{3}}\frac{m_P^3}{|\xi|^2\varphi_I}\frac{\partial}{i\partial(\delta\varphi)}, \tag{6.50}$$

$$\hat{c}_- = \frac{1+\delta}{2\lambda\pi}\frac{m_P^3}{\varphi_I^3}\sqrt{\frac{3}{2\pi}}\,\delta\varphi. \tag{6.51}$$

Now we are ready to find the quantum correlators necessary for the radiation currents. We choose a symmetrized combination of disturbances and their conformal time derivatives, $\Delta Q_{1,2} = (\Delta\widehat{Q}, \Delta\widehat{Q}', \Delta\widehat{Q}'')$, in the definition of the correlator

$$\langle\Delta Q_1 \Delta Q_2\rangle \equiv \frac{1}{2}\int d(\delta\varphi)\,\boldsymbol{\Psi}^*(\delta\varphi)(\Delta\widehat{Q}_1\Delta\widehat{Q}_2 + \Delta\widehat{Q}_2\Delta\widehat{Q}_1)\boldsymbol{\Psi}(\delta\varphi), \tag{6.52}$$

because in the Hermitian operators of quadratic currents (3.9) the products of operator valued disturbances automatically enter in symmetrized form (in view of the symmetry of the 3-vertex function). The further calculation of the correlators and radiation currents is straightforward. However, the general answer that involves the basis functions $\boldsymbol{Q}_{\pm}(\eta)$ for arbitrary η is still very complicated. Therefore, we separately consider the beginning of the inflation epoch, $\eta = 0$, and the late stationary stage of inflation.

7. The onset of inflation

At the onset of inflation the basis functions $\boldsymbol{Q}_{\pm}(\eta)$ have a behaviour (5.38), (5.39). Using it in Eqs. (6.47)–(6.49) with the operators $\hat{c}_{\pm}$ defined by (6.50), (6.51) one easily obtains the initial equal-time correlators with respect to the gaussian state (6.46).

In the leading order of the slow roll expansion those correlators that do not involve derivatives (potential type correlators) read

$$\begin{aligned}
&\langle A^2\rangle_0 = -\langle A\psi\rangle_0 = \langle\psi^2\rangle_0 = \frac{\lambda}{288\pi^2|\xi|^2}\frac{1}{f},\\
&\langle\psi\delta\varphi\rangle_0 = -\langle A\delta\varphi\rangle_0 = \frac{\lambda\varphi_I}{288\pi^2|\xi|^2}\frac{1}{f},\\
&\langle\delta\varphi^2\rangle_0 = \frac{\lambda\varphi_I^2}{288\pi^2|\xi|^2}\frac{1}{f},
\end{aligned} \tag{7.1}$$

while the correlators of conformal time "velocities" (the kinetic type correlators) equal

$$\begin{aligned}
&\langle A'^2\rangle_0 = -\langle A'\psi'\rangle_0 = \langle\psi'^2\rangle_0 = \frac{\lambda}{288\pi^2|\xi|^2} f,\\
&\langle\psi'\delta\varphi'\rangle_0 = -\langle A'\delta\varphi'\rangle_0 = \frac{\lambda\varphi_I}{288\pi^2|\xi|^2} f,\\
&\langle\delta\varphi'^2\rangle_0 = \frac{\lambda\varphi_I^2}{288\pi^2|\xi|^2} f.
\end{aligned} \tag{7.2}$$

As we see, these two groups of correlators differ by the power of a special parameter f which is inverse proportional to the square of quantum dispersion of the inflaton field

$$f \equiv \frac{\lambda}{48\pi^2|\xi|^2}\frac{1}{\kappa\Delta^2} = \left(\frac{\boldsymbol{A}}{16\pi^2}\right)^2\frac{|\xi|}{|1+\delta|}. \tag{7.3}$$

Such a dependence on f reflects an obvious fact that the kinetic type correlators (or correlators of momenta) in the gaussian state of the form (6.46) are inverse proportional to Δ^2, and thus grow with $\Delta \to 0$, while the potential type correlators (correlators of coordinates) are proportional to Δ^2, and thus decrease with the narrowing of the gaussian peak.

The calculation of mixed correlators with one or two derivatives of the form

$$\langle\Delta Q\Delta Q'\rangle_0 = \langle\Delta Q\Delta Q\rangle_0\mathrm{O}(\epsilon^2), \qquad \langle\Delta Q\Delta Q''\rangle_0 = \langle\Delta Q\Delta Q\rangle_0\mathrm{O}(\epsilon) \tag{7.4}$$

shows that they belong to the subleading order in the slow roll parameter (6.11). Finally, the additional correlators with three derivatives, which arise in the calculation of dj_N^q/dt

in the quantum rolling force (6.39), express as

$$\begin{aligned}
&\langle\psi''\delta\varphi'\rangle_0 = \langle\psi'\delta\varphi''\rangle_0 = -2\mathcal{H}\langle\psi'\delta\varphi'\rangle_0,\\
&\langle\delta\varphi'\delta\varphi''\rangle_0 = -2\mathcal{H}\big\langle\delta\varphi'^2\big\rangle_0,\\
&\langle\psi'\psi''\rangle_0 = -2\mathcal{H}\big\langle\psi'^2\big\rangle_0, \quad \eta\to 0.
\end{aligned} \tag{7.5}$$

Although they tend to zero in view of $\mathcal{H}(\eta)\to 0$ at $\eta\to 0$, their contribution to the quantum rolling force is nontrivial because in (6.35) they are divided by $\dot a/a = \mathcal{H}/a$.

Let us now go over to the calculation of radiation currents at $\eta = 0$. Within the slow roll approximation, $m_P^2/|\xi|\varphi^2 \ll 1$, $|\xi|\gg 1$, and in view of a particular form of classical coefficient functions $V(\varphi), U(\varphi)$, the quadratic currents (6.27), (6.28) are dominated by the following expressions involving both the potential and kinetic terms

$$j_N^q(0) = \frac{\lambda\varphi^4}{4}\left[-2\big\langle\psi^2\big\rangle + \frac{10}{\varphi}\langle\psi\delta\varphi\rangle - \frac{15}{\varphi^2}\big\langle\delta\varphi^2\big\rangle + \big\langle\psi'^2\big\rangle - \frac{2}{\varphi}\langle\psi'\delta\varphi'\rangle\right], \tag{7.6}$$

$$j_\varphi^q(0) = \lambda\varphi^3\left[4\langle A\psi\rangle + \frac{7}{\varphi}\langle\psi\delta\varphi\rangle + \frac{1}{2}\langle A'\psi'\rangle\right]. \tag{7.7}$$

On using the tables of correlators above, the radiation currents, contributing to the quantum rolling force, take the following final form

$$j_N(0) = \frac{\lambda\varphi_I^4}{4}\frac{\lambda}{96\pi^2|\xi|^2}\left(\frac{1}{f} - \frac{1}{3}f\right), \tag{7.8}$$

$$j_\varphi(0) = \lambda\varphi_I^3\frac{\lambda}{96\pi^2|\xi|^2}\left(\frac{1}{f} - \frac{1}{6}f\right), \tag{7.9}$$

$$\frac{a}{2\dot a}\frac{d\,j_N}{dt}(0) = 0. \tag{7.10}$$

These quantities are strongly suppressed as compared to their classical values, $-j_N = V(\varphi_I) \simeq \lambda\varphi_I^4/4$, $-j_\varphi = V_\varphi(\varphi_I) \simeq \lambda\varphi_I^3$ by a very small factor $\lambda/|\xi|^2 \sim \Delta T^2/T^2 \sim 10^{-10}$ related to the CMBR anisotropy. Their sign crucially depends on the magnitude of the parameter f, (7.3), which in our model is likely to be very big, $f \gg 1$. This follows from the estimate $N \geqslant 60$ on the e-folding number (2.19) and the value of $|\xi| \sim 10^4$ [27]. In this case, the terms proportional to $f \sim 1/\varDelta^2$, generated by the kinetic terms of the radiation currents, $\langle\Delta Q'\Delta Q'\rangle$, dominate and, in particular, result in

$$\varepsilon^q(0) = -j_N(0) \simeq \frac{\lambda^2|1+\delta|}{18|\xi|^3}\frac{m_P^4}{(16\pi^2)^2} \ll m_P^4. \tag{7.11}$$

Interestingly, the sign of the quantum rolling force due to the homogeneous mode is independent of the magnitude of f, because in the leading order of the slow-roll expansion the contributions of potential terms, $\langle\Delta Q\Delta Q\rangle \sim 1/f$, completely cancel out

$$F^q(0) = \frac{U j_\varphi^q(0) - 2U_\varphi j_N^q(0)}{U + 3U_\varphi^2} \simeq \frac{\lambda\varphi_I^3}{36}\frac{\lambda}{96\pi^2|\xi|^3} f > 0. \tag{7.12}$$

In view of the expressions for f and φ_I, (2.10), the magnitude of this force is again much smaller than its classical counterpart

$$F^q(0) = \frac{\lambda m_P^2}{48\pi|\xi|^3}\varphi_I \frac{\lambda}{144\pi^2}\frac{\boldsymbol{A}}{16\pi^2} \simeq \left|F^{\text{class}}(0)\right| \frac{\lambda}{144\pi^2|\xi|}\frac{\boldsymbol{A}}{16\pi^2|1+\delta|} \\ \ll \left|F^{\text{class}}(0)\right|. \tag{7.13}$$

Therefore, for the tunneling state it gives a negligible contribution to the effective force (6.43). For the no-boundary state, the initial effective force (6.42) vanishes, but the only effect that the positive $F^q(0)$ can produce in this case is that it shifts the equilibrium point from φ_I to slightly higher value of the inflaton φ_*, $F^q(0) + F_{\text{NB}}^{\text{eff}}(\varphi_*) = 0$, at which again the system will undergo endless inflation.

8. Late stage of inflation

At late stationary stage of inflation the dynamics of the classical background can be approximated by the ansatz

$$a = \frac{1}{H(\varphi)}\cosh\bigl[H(\varphi)t\bigr], \qquad \varphi \simeq \varphi_I, \tag{8.1}$$

$$H^2(\varphi) = \frac{V(\varphi)}{6U(\varphi)} \simeq \frac{\lambda\varphi^2}{12|\xi|} \tag{8.2}$$

with the Hubble constant $H(\varphi)$ approximately linear in φ. In the Einstein frame, it looks similar with the Hubble constant which is practically independent of the inflaton $\overline{H}^2(\bar\varphi) = 8\pi\,\overline{V}(\bar\varphi)/3m_P^2 \simeq \lambda m_P^2/96\pi|\xi|^2$, the cosmic time parameters being related in both frames by $\bar t \simeq t\sqrt{8\pi|\xi|}\varphi/m_P$.

The transition period between the onset of inflation and its steady stage can be described by solving the inflaton equation with the approximately constant rolling force F and the friction term based on the ansatz (8.1) for $\dot a/a$

$$\ddot\varphi + 3H\tanh(Ht)\dot\varphi - F = 0, \\ \varphi(0) = \varphi_I, \qquad \dot\varphi(0) = 0. \tag{8.3}$$

For late times, $Ht \gg 1$ (but not so late that the inflaton field evolves too far from its initial value), the exact solution to this equation

$$\varphi(t) = \varphi_I + \frac{F}{3H^2}\ln(\cosh Ht) + \frac{F}{3H^2}\tanh^2(Ht) \tag{8.4}$$

reads as an almost linear function of t

$$\varphi(t) = \varphi_I + \frac{F}{3H}t + \frac{F}{3H^2}(1-\ln 2) + \mathrm{O}\bigl(e^{-2Ht}\bigr). \tag{8.5}$$

This behaviour corresponds to neglecting the $\ddot\varphi$ term in the inflaton equation of motion and solving it for $\dot\varphi$, $\dot\varphi \simeq F/3H$. In our model with the classical rolling force (2.18), $\dot\varphi \simeq -4\varphi H\epsilon/3 \ll H\varphi$ with $\epsilon \sim m_P^2/|\xi|\varphi^2$ — the slow roll smallness parameter (6.11). Thus, in the lowest order of the slow roll approximation the inflaton field remains constant.

Let us study the behaviour of the basis functions $\boldsymbol{Q}_{\pm}(\eta)$ for $Ht \gg 1$. To begin with, note that for late times corresponding to exponentially large scale factor the potential terms in the wave equation for $\boldsymbol{Q}_{\pm}(\eta)$, (5.37), can be discarded. The first one, $\kappa\bar{\varphi}'^2/2 = \mathrm{O}(\epsilon^2)$, is small in view of the slow roll regime and the second one, the spatial curvature term 3, is small compared to the kinetic terms growing with a, $d^2/d\eta^2 \sim a^2 d^2/dt^2$. Therefore, at late times this equation simplifies to

$$\left(-\frac{d^2}{d\eta^2} + 2\frac{\bar{\varphi}''}{\bar{\varphi}'}\frac{d}{d\eta}\right)\boldsymbol{Q}_{\pm}(\eta) = 0, \tag{8.6}$$

and has two explicit solutions $\boldsymbol{Q}_{\pm}$

$$\boldsymbol{Q}_{-}(\eta) = 1, \tag{8.7}$$

$$\boldsymbol{Q}_{+}(\eta) = N_{+}\int_0^{\eta} d\tilde{\eta}\,\bar{\varphi}'^2(\tilde{\eta}), \qquad N_{+} \simeq \left(\frac{8\pi|\xi|\varphi_I^2}{m_P^2}\right)^2 \frac{\pi}{m_P^2(1+\delta)^2}, \tag{8.8}$$

compatible with the expansions (5.38), (5.39) at early times $\eta \to 0$. (The mismatch between the constant function (8.7) and Eq. (5.39) has a simple explanation: the term $-3\eta^2/2$ in $\boldsymbol{Q}_{-}$ of Eq. (5.39) is induced by the curvature term which we discard at late times, while the $O(\eta^3)$ corrections are due to the perturbation $\kappa\bar{\varphi}'^2/2$.) In terms of the cosmic time in the original frame, $\boldsymbol{Q}_{+}$ represents a well known growing mode [43] which for late times in the slow roll approximation reads as

$$\boldsymbol{Q}_{+} = \frac{1}{3}\sinh Ht. \tag{8.9}$$

Thus at late times the operator of the invariant cosmological perturbation is dominated by the growing mode

$$\widehat{\boldsymbol{Q}}(t) = \hat{c}_{+}\boldsymbol{Q}_{+}(t) + \hat{c}_{-}\boldsymbol{Q}_{-}(t) \simeq \hat{c}_{+}\boldsymbol{Q}_{+}(t), \quad Ht \gg 1, \tag{8.10}$$

and in the Newton gauge all the minisuperspace perturbations express in terms of one operator $\hat{c}_{+}$ defined by the momentum $\hat{p}_{\delta\varphi}$, (6.50),

$$\begin{aligned} \bar{A} = \bar{\psi} &= \frac{\kappa\boldsymbol{Q}}{2\bar{a}(t)} \simeq \sqrt{\frac{\pi\lambda}{54}}\,\frac{\hat{c}_{+}}{m_P|\xi|}, \\ \delta\bar{\varphi} &= \frac{\dot{\boldsymbol{Q}}}{\bar{a}(t)\dot{\bar{\varphi}}(t)} \simeq -\frac{\sqrt{2\lambda}}{3}\,\frac{\pi\varphi^2}{m_P^2(1+\delta)}\hat{c}_{+}. \end{aligned} \tag{8.11}$$

Since $\delta\bar{\varphi}$ contains $\dot{\bar{\varphi}}$ in the denominator, all the other perturbations in the minimal frame are much smaller in magnitude, $(\bar{A}, \bar{\psi}) \sim O(\epsilon)\delta\bar{\varphi}/m_P \ll \delta\bar{\varphi}/m_P$. Therefore, in view of Eqs. (6.16)–(6.18) the perturbations in the nonminimal frame read

$$A \simeq -\psi \simeq \sqrt{\frac{4\pi}{3}}\,\frac{\delta\bar{\varphi}}{m_P}, \qquad \delta\varphi \simeq \sqrt{\frac{4\pi}{3}}\,\frac{\varphi}{m_P}\delta\bar{\varphi}, \quad Ht \gg 1. \tag{8.12}$$

Another important property of the perturbations in both frames is that to the leading order in slow roll they are constant in time for $Ht \gg 1$. This follows from Eqs. (8.11) containing the exponentially growing functions of time in both of its numerator and denominator (respectively $\boldsymbol{Q}$ and $\bar{a}$). As a result, the time derivatives of perturbations belong to the subleading order of the slow roll expansion, $\Delta\dot{Q} \equiv (\dot{A}, \dot{\psi}, \delta\dot{\varphi}) = \mathrm{O}(\epsilon)\Delta Q$.

Thus, we arrive at the following list of correlators at late times. The potential type correlators read

$$\begin{aligned}
&\langle A^2\rangle = \langle\psi^2\rangle = -\langle A\psi\rangle = \frac{\lambda}{2592\pi^2|\xi|^2} f,\\
&\langle\psi\delta\varphi\rangle = -\langle A\delta\varphi\rangle = \frac{\lambda\varphi}{2592\pi^2|\xi|^2} f,\\
&\langle\delta\varphi^2\rangle = \frac{\lambda\varphi^2}{2592\pi^2|\xi|^2} f,
\end{aligned} \tag{8.13}$$

where the parameter f is given by Eq. (7.3), while the kinetic type correlators are negligibly small

$$\begin{aligned}
&\langle\Delta Q\Delta Q'\rangle = \mathrm{O}(\epsilon)aH\langle\Delta Q\Delta Q\rangle,\\
&\langle\Delta Q'\Delta Q'\rangle = \mathrm{O}(\epsilon^2)(aH)^2\langle\Delta Q\Delta Q\rangle,\\
&\langle\Delta Q\Delta Q''\rangle = \mathrm{O}(\epsilon)(aH)^2\langle\Delta Q\Delta Q\rangle,\\
&\langle\Delta Q'\Delta Q''\rangle = \mathrm{O}(\epsilon^2)(aH)^3\langle\Delta Q\Delta Q\rangle.
\end{aligned} \tag{8.14}$$

In view of these relations, the terms that give the leading contribution to radiation currents are exhausted by a small fraction of terms in Eqs. (6.27), (6.28). They include only the potential type correlators and read

$$j_N^q = \frac{\lambda\varphi^4}{4}\left[4\langle A^2\rangle + 6\langle A\psi\rangle - \frac{4}{\varphi}\langle A\delta\varphi\rangle + \frac{6}{\varphi}\langle\psi\delta\varphi\rangle - \frac{5}{\varphi^2}\langle\delta\varphi^2\rangle\right], \tag{8.15}$$

$$j_\varphi^q = \lambda\varphi^3\left[2\langle A^2\rangle + 6\langle A\psi\rangle - \frac{8}{\varphi}\langle A\delta\varphi\rangle + \frac{6}{\varphi}\langle\psi\delta\varphi\rangle - \frac{3}{\varphi^2}\langle\delta\varphi^2\rangle\right]. \tag{8.16}$$

The resulting radiation currents, thus, equal

$$j_N^q = \frac{\lambda\varphi^4}{4}\frac{\lambda}{864\pi^2|\xi|^2} f, \tag{8.17}$$

$$j_\varphi^q = \frac{\lambda\varphi^3}{2}\frac{\lambda}{864\pi^2|\xi|^2} f, \quad Ht \gg 1. \tag{8.18}$$

Similarly to the onset of inflation, Eqs. (7.8), (7.9), they are strongly suppressed relative to the classical values, $-V(\varphi) \simeq -\lambda\varphi^4/4$ and $-V_\varphi(\varphi) \simeq -\lambda\varphi^3$ by the factor $\lambda/|\xi|^2 \sim 10^{-10}$. In absolute units, the energy density of the quantum mechanical mode is given by

$$\varepsilon^q = -j_N^q \simeq -\frac{\lambda^2|1+\delta|}{54|\xi|^3}\frac{m_P^4}{(16\pi^2)^2}, \quad |\varepsilon^q| \ll m_P^4. \tag{8.19}$$

Note that, in contrast to the onset of inflation, this energy is negative. Apparently, this is a manifestation of the ghost nature of the invariant physical mode $\boldsymbol{q}$, whose kinetic term enters the action (5.18) with the wrong sign. In the slow roll approximation with negligible $\dot{\varphi}$ a constant energy density (8.19) in view of Eq. (3.30) — the conservation law for radiation current — generates the effective equation of state of the inflaton excitation mode

$$\varepsilon^q + p^q = 0, \qquad \varepsilon^q < 0, \qquad Ht \gg 1, \tag{8.20}$$

which in the absence of other sources would maintain the anti-de Sitter spacetime. Similar equation of state at the onset of inflation corresponds to the de Sitter case, so that the inflaton quantum excitation undergoes a sort of phase transition reversing the sign of its energy density.

In view of the relation between the radiation currents, $j^q_N = \varphi j_\varphi/4 + \mathrm{O}(\epsilon)$, and approximately constant value of j^q_N, $dj^q_N/dt = \mathrm{O}(\epsilon)$, the quantum rolling force in Eq. (6.35),

$$F^q \simeq \frac{1}{6|\xi|\varphi}\left(\varphi j^q_\varphi - 4j^q_N - \frac{1}{2H}\frac{d\,j^q_N}{dt}\right) = \mathrm{O}(\epsilon), \tag{8.21}$$

vanishes in the leading order of the slow roll expansion. Thus, similarly to the onset of inflation, at late times of inflation epoch F^q does not qualitatively change the cosmological evolution.

9. Conclusions

We have developed a general framework for effective equations of inflationary dynamics in quantum cosmology and for their quantum Cauchy problem with no-boundary and tunneling quantum states. This framework combines the Euclidean effective action method and the method of direct quantum averaging for calculations of two distinctly different parts of radiation currents — contributions of the field theoretical and quantum mechanical (minisuperspace) sectors of the system. We focus on the latter and show that its calculation is based on explicit physical reduction for the spatially homogeneous cosmological perturbation. Because of the ghost nature of this perturbation, its effect is not related to the conventional analytic continuation from the Euclidean spacetime. Rather, in the model of strongly coupled nonminimal inflaton it originates from the quasi-gaussian state which incorporates the tree-level and one-loop effects on the de Sitter instanton. It is, thus, irrelevant to the de Sitter invariant Euclidean vacuum and cannot be obtained by analytic continuation from the Euclidean section of spacetime. This means that the universality of analytic continuation methods of [44] should not be overestimated — they apply to spatially inhomogeneous, particle like excitation but may fail for minisuperspace cosmological modes.

Unfortunately, the dynamical contribution of the quantum mechanical mode to effective equations turned out to be disappointingly small — it is strongly dominated by the effective rolling force (contributed on equal footing by the classical term and the one-loop term due to the inhomogeneous modes). The property of its strong suppression by powers of $1/|\xi| \ll 1$ was actually conjectured in [1], and now it is quantitatively confirmed. Thus, the inflaton mode cannot change the dynamical predictions in spatially closed model with strong nonminimal coupling. As a model of the low-energy quantum origin of the Universe only the tunneling state remains observationally justified, because the no-boundary wavefunction generates infinitely long inflationary stage. The role of this mode should not, however, be underestimated, because its effect is model dependent, and might be important in other models generating initial conditions for inflation [45]. Moreover, the quantum inflaton mode simulates the de Sitter and anti-de Sitter effective equations of state, $\varepsilon + p = 0$, respectively at the onset of inflation and at late times. The sign of its energy density contribution can change depending on the balance of the potential and kinetic terms of this ghost mode. Therefore, it is not quite clear at the moment, what can the role of this mode be at post inflationary epoch. A natural question arises if this mode can be responsible for the present day observable acceleration of the Universe [46] as an alternative to quintessence [47] or be capable of inducing de Sitter–anti-de Sitter phase transitions in cosmology? This question is subject to further studies [45].

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